Stellations of Two Cores

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Dedicated to the memory of Magnus Wenninger.

1 Motivation

My fascination with stellations of polyhedra began with two classic works: Wenninger's *Polyhedron Models* [1] and Coxeter, DuVal, Flather, and Petrie's *The Fifty-Nine Icosahedra* [2]. I had the pleasure of an active correspondence with Magnus Wenninger; I would often send him sets of notes I had written about polyhedra and he would offer commentary. His passing in February 2017 inspired me to unearth these notes and organize them a bit more formally. This paper is based on a set of hand-written notes dated September 1994.

2 Introduction

What is a stellation? This question is perhaps as difficult to answer as "What is a polyhedron?" The most usual definition of a stellation of a polyhedron is some symmetric collection of cells which space is divided into by the facial planes of the polyhedron. Miller's rules for defining stellations are referenced in *The Fifty-Nine Icosahedra.* More recently, Hudson and Kingston offered their perspective in the *Intelligencer* [3].

For our purposes, a precise definition is not required. Rather, our aim is to suggest a possible generalization of the current notion of "stellation."

We illustrate with a specific example, the uniform polyhedron sometimes called the *stellated truncated hexahedron,* shown in Figure 1. This nomenclature is unfortunate, since this polyhedron is actually *not* a stellation of any truncated hexahedron, since the eight facial planes containing the triangular faces bound an octahedron which lies entirely in the interior of the cube formed by the facial planes of the six octagrams. This is clear given how deeply the pink triangular faces cut into the interior of the cube.

Yet this polyhedron has six faces lying in the facial planes of a cube and eight faces lying in the facial planes of an octahedron. Thus, we call this polyhedron a *stellation of two cores,* since it is not possible to create as a stellation of a single polyhedron. Of course there may be multiple cores – and in general, we might consider *any* collection of planes in space. But because of the high symmetry of uniform polyhedra, we will confine our attention to stellations of multiple cores, and in this paper, to just two cores – a cube and an octahedron.

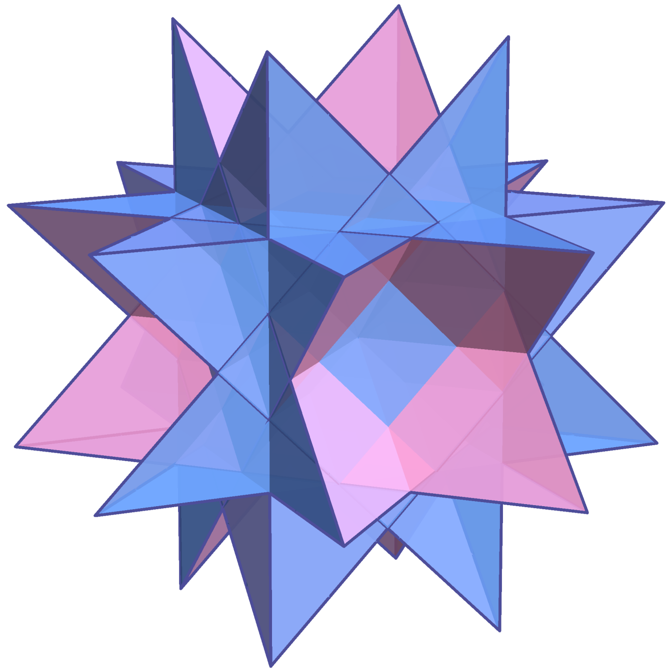

Figure 1: Stellated truncated hexahedron.

It is instructive to describe in some detail an example in two dimensions; the ideas extend naturally to three dimensions. First look at Figure 2(d), where a square and the lines containing the sides (analogues of facial planes in three dimensions) are drawn together with the dual square and the lines containing its sides (shown in red). In three dimensions, the dual of the cube will be the octahedron, so we say "dual square" here to distinguish the squares. We assign the dual square a *scale factor* $\sigma = 1$.

Now imagine shrinking the dual square slightly, together with the lines containing its sides. The plane will topologically be divided into the same number and type (that is, finite or infinite) of cells, although some may now be shaped differently. But when the dual square shrinks by a value of $\sigma = 1/\sqrt{2}$ (see Figure 2(c)), some of the cells degenerate to points. We call these values where of σ where the topology changes *transitional values* of σ.

As σ continues to shrink, new cells are now formed (see Figure 2(b)). The topology of the cells does not change until the limiting case $\sigma = 0$, illustrated in Figure 2(a).

As σ grows larger than one, the topology of the cells remains the same until the transitional value $\sigma = \sqrt{2}$, as in Figure 2(e). (Note that this looks like a rotated and enlarged version of the $\sigma = 1/\sqrt{2}$ case, but in three dimensions, this will not be case as the dual of the cube is an octahedron.) As σ gets larger (see Figure 2(f)), the topology remains the same until the limiting case $\sigma = \infty$ (Figure 2(g)), where lines containing opposite sides of the square are now coincident.

Note that only one of these diagrams (Figure 2(d)) represents a stellation of a *single* polygon, namely a convex octagon. The other six cases cannot be so described. But interesting

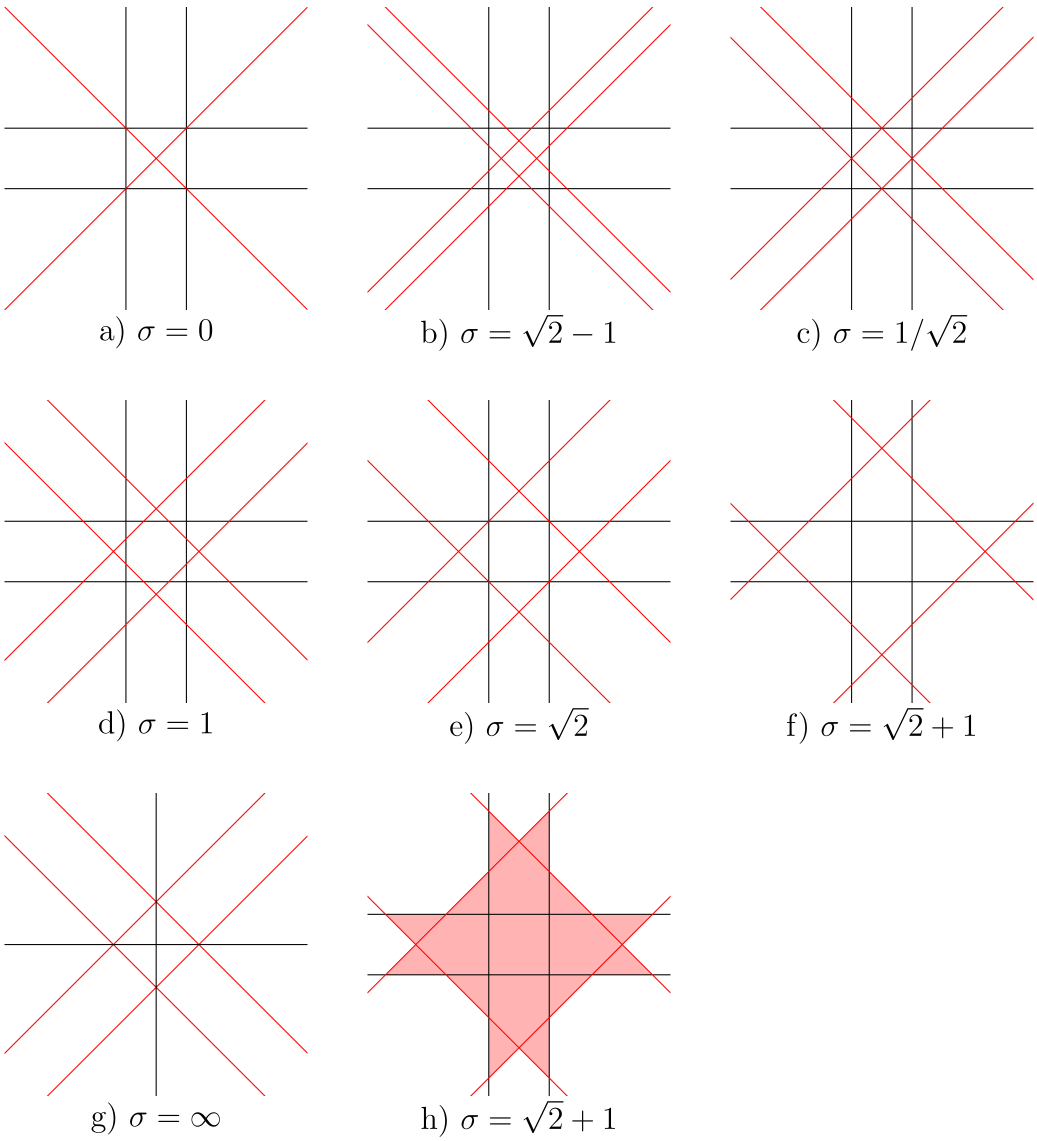

Figure 2: Topologically distinct stellations of two cores.

polygons, such as the one shaded in Figure 2(h), are generated even though they cannot be considered as stellations of a single polygon.

3 Jumping to Three Dimensions

We now consider a three-dimensional analogue of the previous discussion. We begin with a cube **C** and its dual octahedron **O**, where we define the dual in such a way that the edges of **C** and **O** perpendicularly bisect each other. How do the facial planes of these polyhedra divide space?

First consider how the facial planes of the cube divide space. First, we have the cube $\mathbf{C}_0$. Next, we have the the six prismatic infinite regions over the faces, which we collectively denote as $\mathbf{C}_1$. The twelve infinite regions over the edges sharing unbounded faces with the $\mathbf{C}_1$ are denoted by $\mathbf{C}_2$. Finally, we denote by $\mathbf{C}_3$ the eight octants on top of the $\mathbf{C}_2$ which meet the cube at its vertices. This gives the cell adjacency diagram shown in Figure 3. We note that although infinite regions are typically not included in cell adjacency diagrams (which we also refer to as *stellation diagrams*), consideration of Figure 2(b), for example, shows that two infinite regions – one from the square, and one from the dual square – may intersect in a bounded region. So we must consider the complete decomposition of space by the facial planes. For simplicity, we refer to *the* cell $\mathbf{C}_1$, for example, even though $\mathbf{C}_1$ is actually a collection of cells.

$$\begin{array}{c}\mathbf{C}_3\\ |\\ \mathbf{C}_2\\ |\\ \mathbf{C}_1\\ |\\ \mathbf{C}_0\end{array}$$

Figure 3: Infinite stellation diagram for the cube.

We now consider how the facial planes of the octahedron divide space. We have $\mathbf{O}_0$, the octahedron itself. Above the triangular faces are the eight tetrahedra $\mathbf{O}_1$ of the stella octangula. The infinite regions are most easily described by Figure 4, which shows the eight facial planes of **O** dividing up two spheres containing $\mathbf{O}_0$ at its center. The point at the center of each image is a vertex of the stella octangula; the sphere used on the right is twice the radius of the one on the left. This gives two different perspectives from which to understand the geometry of the remaining cells.

There are twelve regions $\mathbf{O}_2$ with rhombic cross-sections above the edges of the octahedron (and sharing two triangular faces with the stella octangula). On top of these, with trapezoidal cross-sections, are the 24 cells $\mathbf{O}_3$. There are two types of infinite cells on top of the

$\mathbf{O}_3$: the six square pyramidal regions $\mathbf{O}_4$ (opposite the vertices of the octahedron) and the eight triangular pyramidal regions $\mathbf{O}_5$ (opposite the vertices of the stella octangula). The corresponding infinite stellation diagram is shown in Figure 5.

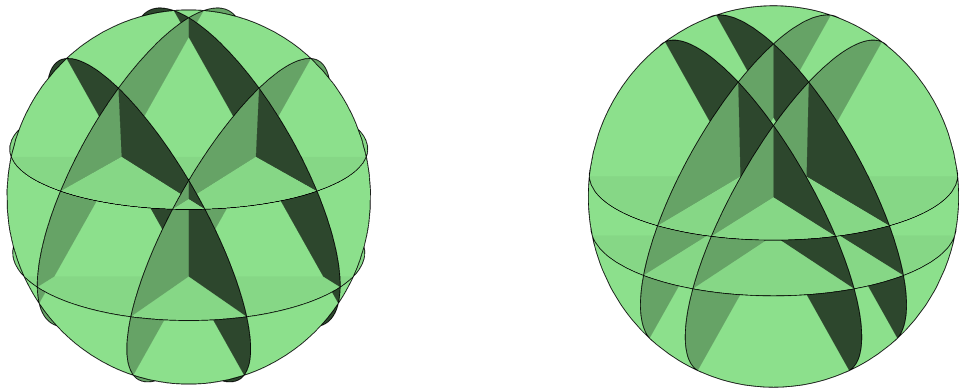

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Figure 4: Decomposition of space by octahedral facial planes.

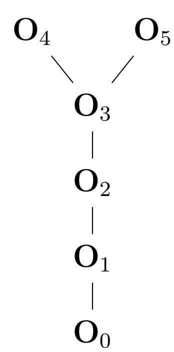

Figure 5: Infinite stellation diagram for the octahedron.

To continue the analysis as illustrated in two dimensions in Figure 2, we consider the facial planes of the cube $\mathbf{C}_0$, and the facial planes of the scaled $\sigma\mathbf{O}_0$, where $0 \le \sigma \le \infty$. For each region $\mathbf{C}_i \cap \sigma\mathbf{O}_j$, where $0 \le i \le 3$ and $0 \le j \le 5$, we wish to determine if the region is:

1. empty;
2. degenerate – that is, having dimension less than three;
3. bounded and non-degenerate – that is, enclosing a nonzero finite volume; for simplicity, we will refer to such regions as *finite*;
4. infinite – that is, enclosing an infinite volume.

Essentially, we need to determine the finite cells for various σ, since stellations of these two cores will necessarily include such cells.

Some abbreviated notation is convenient. For a given σ, we label the cell $\mathbf{C}_i \cap \sigma\mathbf{O}_j$ as ij. What we will eventually accomplish is a classification of all uniform polyhedra with 14 facial planes (six hexahedral and eight octahedral) as stellations of two cores. For example, when $\sigma = (\sqrt{2}-1)/2$, the stellated truncated hexahedron is in fact a stellation of $\mathbf{C}$ and $\sigma\mathbf{O}$, with the outwardly visible cells being 04 and 15.

4 Transitional Values of σ

First, it is necessary to find the transitional values of σ. Certainly $\sigma = 0$ and $\sigma = \infty$ are *always* transitional values regardless of the two cores, since the topology clearly changes – for one of the cores, parallel opposite planes are now coincident (see the two-dimensional analogues in Figure 2(a),(g)).

We begin our analysis with $\sigma = 0$, and then imagine the core octahedron slowly expanding until previously nondegenerate cells become degenerate. As the octahedron expands, note that the cell 04 is finite, but becomes degenerate when the vertices of the octahedron reach the centers of the cube faces. It is not hard to show that this occurs at the transitional value $\sigma = 1/2$ – just observe that the height of $\mathbf{O}$ must be twice the height of $\mathbf{C}$ because of the way the edges of the octahedron are bisected by the edges of the cube. See Figure 6, where we are looking at the black cube face on; the red diamond is a cross-section of $\mathbf{O}$, the blue diamond is a cross-section of $\frac{3}{2}\mathbf{O}$, and the small orange diamond is a cross-section of $\frac{1}{2}\mathbf{O}$.

As the octahedron continues to expand, a new topology is created. It remains unchanged until the edges of the octahedron hit the edges of the cube and previously finite cells become degenerate – but this is just the transitional value $\sigma = 1$, since this occurs precisely as the octahedron becomes the exact dual of the cube.

The next transitional value of σ occurs when the octahedron expands to the point when the vertices of the cube are centers of the octahedral faces. This must occur when $\sigma = 3/2$; just consider the fact that the barycenter of a triangle divides medians in the ratio $2 : 1$. The dashed blue line in Figure 6 is the projection of such a median.

Once the octahedron expands beyond the cube, the topology remains the same until $\sigma = \infty$.

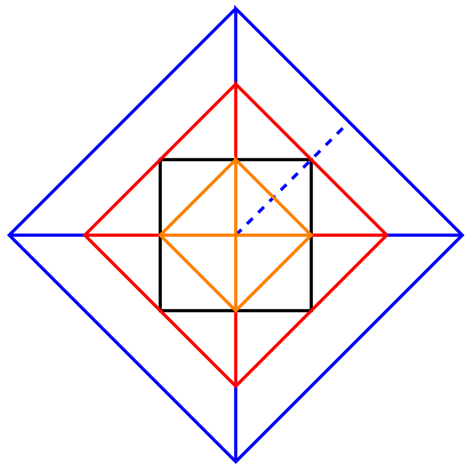

Figure 6: Cross-section of cube with scaled octahedra.

5 Cell Status

We now wish to determine the status of the cells ij for each possible value of σ. For a given σ, some of the cells will be finite, and it is these which we wish to consider when creating a cell diagram for stellations.

We will consider $i = 0$, and leave the other cases to the reader as the analysis is similar. Beginning with 00, since the centers of $\mathbf{C}_0$ and $\mathbf{O}_0$ are coincident, we see that 00 is finite when $0 < \sigma < \infty$. When $\sigma = 0$ or $\sigma = \infty$, 00 degenerates to a single point.

When $\sigma = 0$, the cell 01 degenerates to a point. When $\sigma = \frac{3}{2}$, $\mathbf{C}_0$ is inscribed in $\mathbf{O}_0$, and hence 01 also degenerates to a point. Between these transitional values, 01 is finite, and is empty when $\sigma > \frac{3}{2}$.

We see that 02 is degenerate when when $\sigma = 0$. When $\sigma = 1$, $\mathbf{O}_2$ meets $\mathbf{C}_0$ at the midpoint of an edge of $\mathbf{C}_0$, and hence is degenerate. Thus, we see that 02 is finite when $0 < \sigma < 1$, and empty when $\sigma > 1$.

The cell 03 is degenerate when $\sigma = 0$. Note that the stella octangula is inscribed in $\mathbf{C}_0$ when $\sigma = \frac{1}{2}$, and hence 03 is degenerate in this case. So 02 is finite between these two transitional values, and empty when $\sigma > \frac{1}{2}$.

When $\sigma = 0$, the cell 04 is finite; in fact, a typical 04 region is a half-octahedron whose vertices are the center of $\mathbf{C}_0$ and the four midpoints of the edges of a face of $\mathbf{C}_0$. Again, as the stella octangula is inscribed in the cube when $\sigma = \frac{1}{2}$, 04 is degenerate in this case. Thus, 04 is finite when $0 < \sigma < \frac{1}{2}$, and empty when $\sigma > \frac{1}{2}$.

Finally, we consider the cell 05. When $\sigma = 0$, these regions are those remaining after the six half-octahedra (see the previous case) are removed from $\mathbf{C}_0$. The analysis remains the same as in this case, so that 05 is finite when $0 \leq \sigma < \frac{1}{2}$, degenerate when $\sigma = \frac{1}{2}$, and empty when $\sigma > \frac{1}{2}$.

These data, and data for all the other cells, are given in Table 1. Columns are headed either by transitional values of σ or ranges for σ. Empty cells are represented by an empty entry, degenerate cells by "•," finite cells by "BND" (for bounded and non-degenerate), and infinite cells by "∞."

6 Cell Adjacency Diagrams

How can we use this information to create cell adjacency diagrams? We illustrate with an example, which although simple, illustrates the general case. Consider the case $\sigma = 1$, illustrated in Figure 7, where cell 00 is the cuboctahedron. Two of the square pyramidal cells 10 are shown, slightly shifted for the illustration. What is happening is that the cell $\mathbf{O}_0$ is being sliced by the facial planes of $\mathbf{C}_0$. These facial planes divide $\mathbf{O}_0$ into two types of regions: those parts of $\mathbf{O}_0$ belonging to $\mathbf{C}_0$ and those parts belonging to $\mathbf{C}_1$. Of course $\mathbf{C}_1$ is on top of $\mathbf{C}_0$, and hence the cells in $\mathbf{C}_1 \cap \mathbf{O}_0$ must be on top of $\mathbf{C}_0 \cap \mathbf{O}_0$.

We generalize to the following heuristics for cell adjacency.

1. Cell ki lies on top of cell ji only if $\mathbf{C}_k$ lies on top of $\mathbf{C}_j$;
2. Cell ik lies on top of cell ij only if $\mathbf{O}_k$ lies on top of $\mathbf{O}_j$;
3. When $i \neq k$ and $j \neq l$, cells ij and kl cannot be adjacent.

Next, we create a "master diagram," shown in Figure 8. The heuristics imply that this diagram includes all possible adjacencies between cells. All that is necessary is to determine which of these cells are finite for a given σ, and then read off the adjacencies from the diagram.

We now investigate the value of σ which generates Figure 1. We assume that the edge length of $\mathbf{C}_0$ is 2. It is easy to see that the vertices of Figure 1 are the vertices of a small rhombicuboctahedron, whose vertices are $(1, 1, 1+\sqrt{2})$ together will all possible permutations and changes of signs in the coordinates. A straightforward calculation shows that the midpoints

Cell	0	$(0, \frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2}, 1)$	1	$(1, \frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2}, \infty)$	∞
00	$\bullet$	BND	BND	BND	BND	BND	BND	BND	$\bullet$
01	$\bullet$	BND	BND	BND	BND	BND	$\bullet$		
02	$\bullet$	BND	BND	BND	$\bullet$				
03	$\bullet$	BND	$\bullet$						
04	BND	BND	$\bullet$						
05	BND	BND	$\bullet$						
10			$\bullet$	BND	BND	BND	BND	BND	$\bullet$
11			$\bullet$	BND	BND	BND	BND	BND	$\bullet$
12	$\bullet$	BND	BND	BND	BND	BND	BND	BND	$\bullet$
13	$\bullet$	BND	BND	BND	BND	BND	BND	BND	$\bullet$
14	∞	∞	∞	∞	∞	∞	∞	∞	$\bullet$
15	BND	BND	$\bullet$						
20					$\bullet$	BND	BND	BND	$\bullet$
21			$\bullet$	BND	BND	BND	BND	BND	$\bullet$
22	$\bullet$	∞	∞	∞	∞	∞	∞	∞	$\bullet$
23	$\bullet$	∞	∞	∞	∞	∞	∞	∞	$\bullet$
24	∞	∞	∞	∞	∞	∞	∞	∞	$\bullet$
25	∞	∞	$\bullet$						
30							$\bullet$	BND	BND
31			$\bullet$	BND	BND	BND	BND	BND	BND
32	∞	∞	∞	∞	∞	∞	∞	∞	∞
33	∞	∞	∞	∞	∞	∞	∞	∞	∞
34	∞	∞	∞	∞	∞	∞	∞	∞	∞
35	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table 1: Cell status of the cells ij, $0 \le i \le 3$, $0 \le j \le 5$.

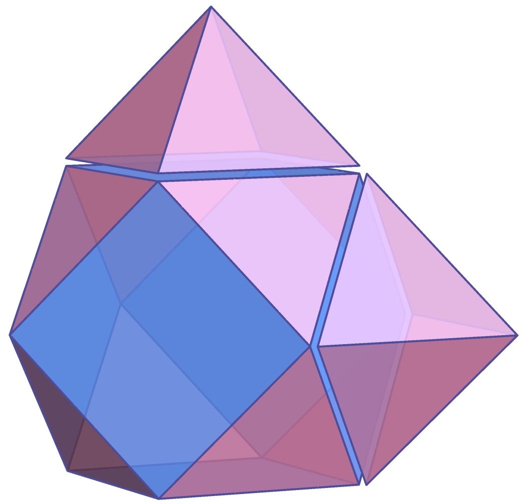

Figure 7: Cell 00 and some of the cells 10.

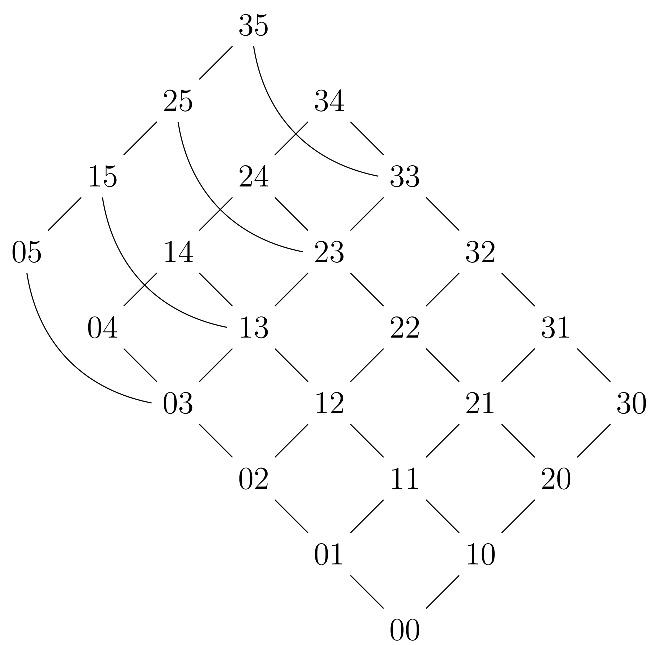

Figure 8: Master stellation diagram.

of the triangular faces of Figure 1 are $\left(\frac{\sqrt{2}-1}{3}, \frac{\sqrt{2}-1}{3}, \frac{\sqrt{2}-1}{3}\right)$, along with all possible changes of sign, so that the distance between opposite triangular faces is $2(\sqrt{2}-1)/\sqrt{3}$. But this means that the distance between opposite faces of the core octahedron is also $2(\sqrt{2}-1)/\sqrt{3}$.

Now if the edge length of $\mathbf{C}_0$ is 2, then the vertices of the dual octahedron are $(0, 0, 2)$, along with all possible permutations and changes of sign. A straightforward calculation shows that the distance between opposite faces on the dual octahedron is $4/\sqrt{3}$. Thus we have

$$\sigma = \frac{2(\sqrt{2}-1)}{\sqrt{3}} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{2}-1}{2} \approx 0.207.$$

We may use this value of σ along with Table 1 and Figure 8 to generate the stellation diagram in Figure 9. Note that we only include those cells from Figure 8 which are listed as finite ("BND") in the second column of Table 1 since $0 < 0.207 < 1/2$.

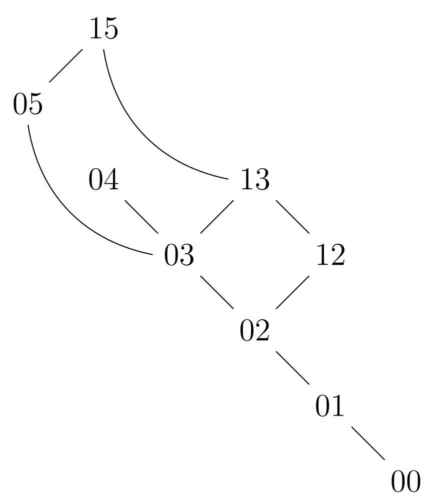

Figure 9: Stellation diagram for $\sigma = \frac{\sqrt{2}-1}{2}$.

The outwardly visible cells of the stellated truncated hexahedron (see Figure 1) are the cells 04 and 15. We remark that for the purposes of this discussion, we need go no further than this. We will simply assume that the stellated truncated hexahedron is solid in that any interior cells not visible are included in the polyhedron. Such stellations are called *fully-supported stellations.*

Figure 9 may be a relatively simple diagram, but it is possible for these diagrams to become enormously complex when the core polyhedra have more faces.

7 Cuboctahedral Uniform Polyhedra

What other uniform polyhedra may be similarly described as stellations of the two cores $\mathbf{C}_0$ and $\mathbf{O}_0$? We might call such polyhedra *cuboctahedral uniform polyhedra.* Trivially, we have the truncated octahedron, cuboctahedron, and truncated cube, corresponding to σ taking on the values $\frac{3}{4}$, 1, and $\frac{\sqrt{2}+1}{2}$, respectively. Of course each of these Archimedean solids corresponds to the cell 00.

The only other nontrivial example is the cubohemioctahedron, where $\sigma = 0$, illustrated in Figure 10. Recall that "hemi" means that the four hexagons pass through the center of the polyhedron. We still consider this a cuboctahedral uniform polyhedron, as there are eight facial planes bounding $0\mathbf{O}$, coincident in pairs.

We summarize these five examples in Table 2. As sometimes names for polyhedra may vary (for example, "great rhombicuboctahedron" and "truncated cuboctahedron" refer to the same polyhedron), the polyhedra are indexed by their Wythoff symbols, which are unambiguous. (Although not necessary here, more information about Wythoff symbols may be found in Coxeter's *Regular Polytopes* [4].)

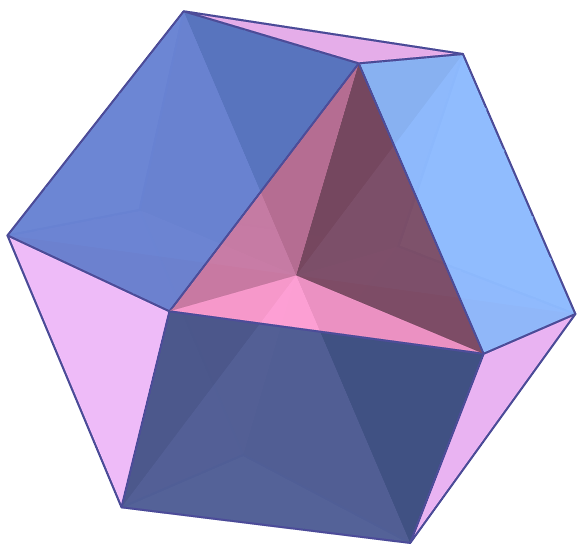

Figure 10: Cubohemioctahedron.

8 Final Remarks

There are 25 other uniform polyhedra which may be described as stellations of two cores. Three are somewhat simpler that those discussed here, while the others are a bit more complex – especially those where one core is the rhombic triacontahedron.

Wythoff symbol	σ	Visible cells	Name
$\frac{4}{3}$ 4 | 3	0	04	Cubohemioctahedron
2 3 | $\frac{4}{3}$	$\dfrac{\sqrt{2}-1}{2}$	04, 15	Stellated truncated hexahedron
2 4 | 3	$\dfrac{3}{4}$	00	Truncated octahedron
2 | 3 4	1	00	Cuboctahedron
2 3 | 4	$\dfrac{\sqrt{2}+1}{2}$	00	Truncated cube

Table 2: Cuboctahedral uniform polyhedra.

My notes of 1994 include similar considerations for those uniform polyhedra which may be considered as stellations of two dodecahedral cores, and those whose cores are an icosahedron and its dual dodecahedron. While such uniform polyhedra are rather more interesting than the cuboctahedral ones, I thought it preferable to illustrate the general idea of stellations of multiple cores with a fairly simple (though nontrivial) example.

Vladimir Bulatov wrote an excellent applet [5][1] which allows the user to specify stellations with multiple cores. It is invaluable for anyone interested in pursuing these ideas further.

References

[1] Magnus J. Wenninger, *Polyhedron Models,* Cambridge, 1971.

[2] H.S.M. Coxeter, P. DuVal, H.T. Flather, and J.F. Petrie, *The Fifty-Nine Icosahedra,* U. Toronto Pr., 1938, (Springer-Verlag reprint, 1982), (Tarquin reprint 1999).

[3] J. L. Hudson and J. G. Kingston,"Stellating Polyhedra," *The Mathematical Intelligencer,* Vol. 10, No. 3, pp. 50–61, 1988.

[4] H.S.M. Coxeter, *Regular Polytopes,* Macmillan, 1963, (Dover reprint, 1973).

[5] Vladimir Bulatov,"An Interactive Creation of Polyhedra Stellations with Various Symmetries," in *Proceedings of Bridges: Mathematical Connections in Art, Music, and Science,* 2001.

[1] The website for the applet referenced in this paper is out of date; the applet is currently accessible via http://bulatov.org/polyhedra/stellation_applet/.