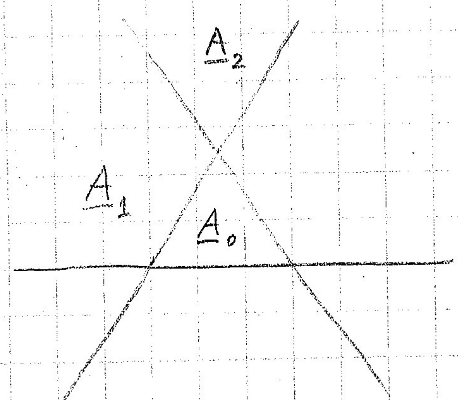
STELLATIONS IN TWO DIMENSIONS

Many of the concepts relevant to a discussion of a general theory of stellations in three dimensions may be illustrated in an analogous discussion of two-dimensional cases. As a result, it is worthwhile to consider a few examples.

Let us first consider an equilateral triangle. By extending the edges of the triangle, we divide space into several regions, some bounded, some unbounded. We may group these regions into various types, each type representing a minimal group of regions having the same symmetry as the triangle. See Figure 1(a)



(a)

(b)

Figure 1

Thus, $\underline{A}_1$ represents the three unbounded regions sharing an edge with $\underline{A}_0$, and $\underline{A}_2$ represents the three unbounded regions opposite the vertices of $\underline{A}_0$. In Figure 1(b) is shown a hierarchy of adjacency,

adjacency being determined by the sharing of an edge, with the original polygon ($\underline{A}_0$ in this case) always being "lowest" in the hierarchy. Such a hierarchy is called a *cell diagram*.

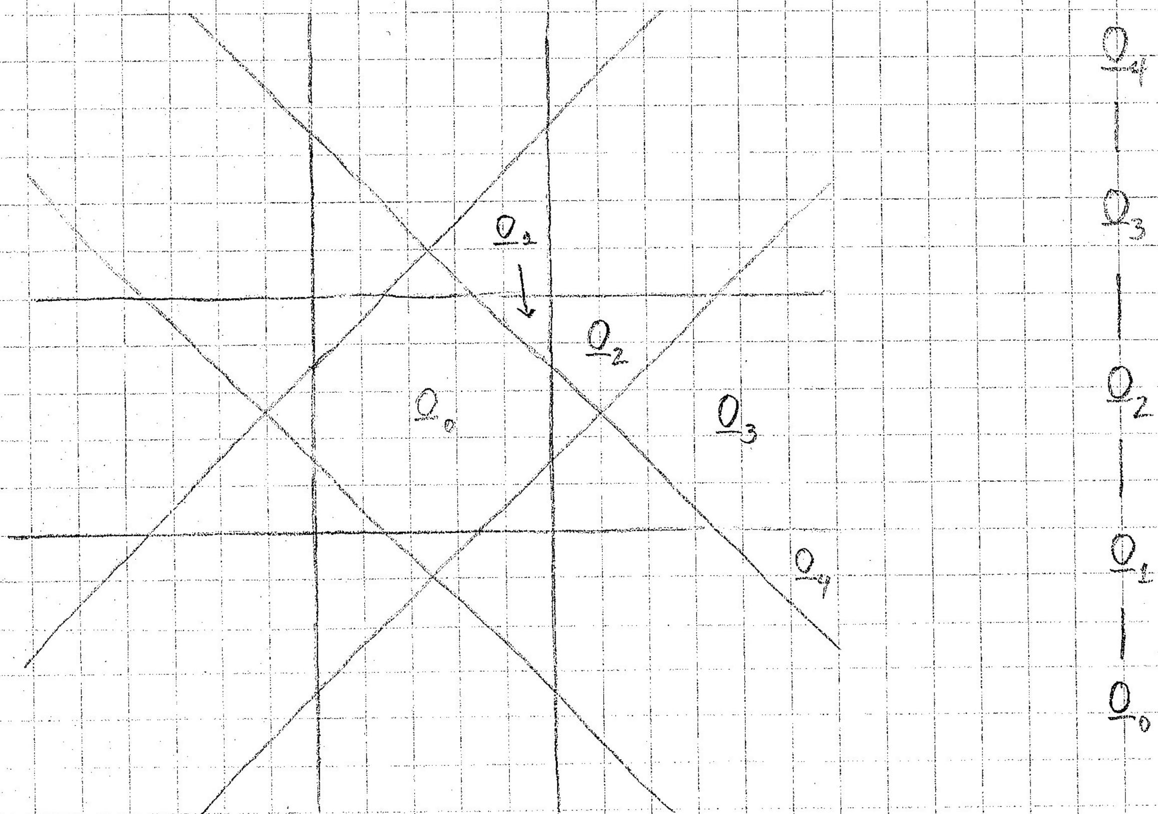


Figure 2

Figure 2 shows the analogous planar decomposition by a regular octagon and the corresponding cell diagram. Note that cells $\underline{O}_0$, $\underline{O}_1$, and $\underline{O}_2$ are bounded. Moreover, $\underline{O}_0$ together with the eight cells of type $\underline{O}_1$ comprise the interior of an octagram (eight-pointed star); adding the eight $\underline{O}_2$ yields a larger "star" still. Such "stars" are called *stellations* of the octagon.

It is worthwhile to note that although in this example and the last the cell diagrams have been "linear", we will see that this is not always the case.

We may look at Figure 2 from another perspective. Although the lines in Figure 2 were constructed by extending the edges of an octagon, we might also describe the diagram by saying that the lines were constructed by extending the edges of two squares, one rotated 45° with respect to the other. Either perspective is valid, but the latter presents new possibilities, which we now proceed to explore.

Let us begin with two equilateral triangles of equal size, but one rotated 60° with respect to the other. Figure 3 shows the corresponding planar decompositions.

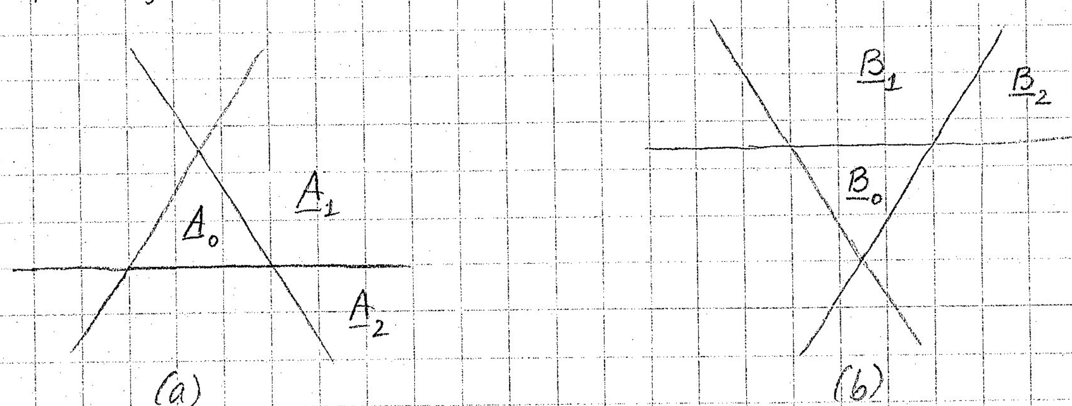


Figure 3

Our next step is to "superimpose" the two diagrams so that the centers of the triangles coincide. The corresponding planar decomposition is given in Figure 4. Note that each region in Figure 4 belongs to one region in Figure 3(a) and one region in

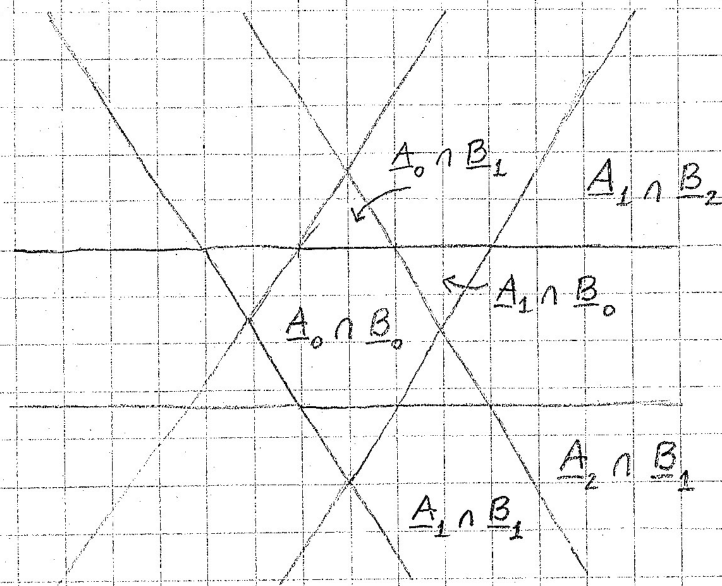
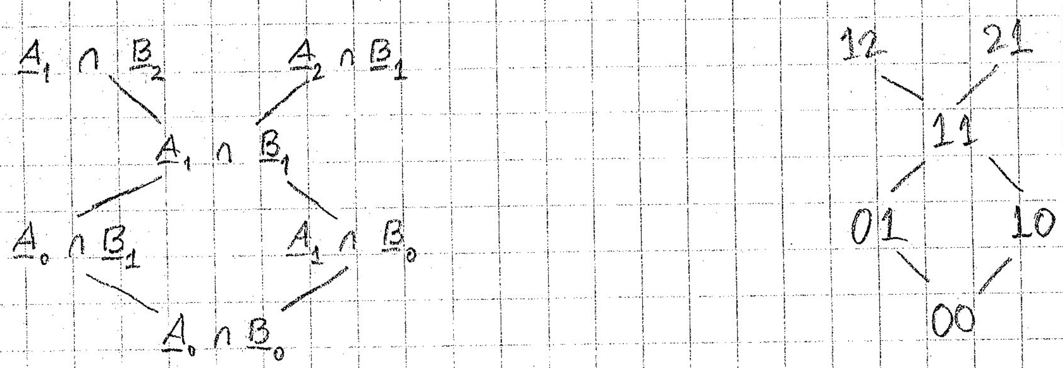


Figure 4

Figure 3(b), and hence may be described as the intersection of these two regions. This is the approach taken in Figure 4. Note that there are three regions of types $\underline{A}_0 \cap \underline{B}_1$, $\underline{A}_1 \cap \underline{B}_0$, $\underline{A}_1 \cap \underline{B}_2$, and $\underline{A}_2 \cap \underline{B}_1$, and six regions of type $\underline{A}_1 \cap \underline{B}_1$. $\underline{A}_0 \cap \underline{B}_0$ is simply a hexagon inscribed in $\underline{A}_0$ and $\underline{B}_0$. Since $\underline{A}_0$ and $\underline{B}_2$ share no common points, $\underline{A}_0 \cap \underline{B}_2$ is empty, and thus not present in our diagram. Likewise, $\underline{A}_2 \cap \underline{B}_0$ and $\underline{A}_2 \cap \underline{B}_2$ are empty. The corresponding cell diagram is given in Figure 5. Note that Figure 5(b) is a "condensed" version of



(a) Figure 5 (b)

Figure 5(a), where each $\underline{A}_i \cap \underline{B}_j$ is represented by its subscripts, ij. This convention will be adopted in order to make the diagrams simpler.

Note that cells 00 and 01 give us the triangle $\underline{A}_0$, 00 and 10 give us the triangle $\underline{B}_0$, and 00, 01, and 10 give us a hexagram, or "Star of David". Each of these may be considered a stellation of the triangles $\underline{A}_0$ and $\underline{B}_0$. We call such stellations "stellations of two cores" since they are obtained by extending the edges of two "core" polygons.

Of course, there is no reason why the triangle $\underline{B}_0$ must be the same size as $\underline{A}_0$; we may produce "variations on a theme" by altering the size of $\underline{B}_0$. So suppose we keep the size of $\underline{A}_0$ the same, but scale $\underline{B}_0$ by a factor of $\frac{3}{2}$. Then the diagram in Figure 4 is transformed into the diagram in Figure 6.

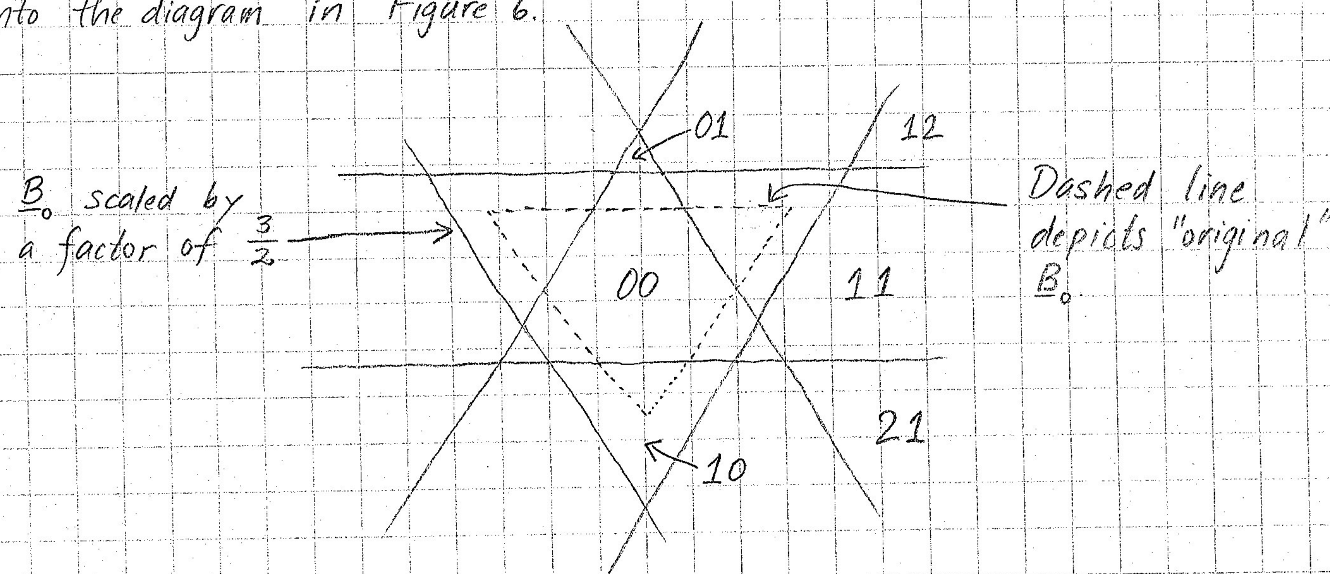


Figure 6

Notice that the cell diagram for Figure 6 is precisely the cell diagram in Figure 5(b); i.e, the one obtained from Figure 4. However, we must not jump to the conclusion that all scale factors for $\underline{B}_0$ result in the same cell diagram. For suppose we consider scaling $\underline{B}_0$ by a factor of 2 (see Figure 7). In this case, the vertices of $\underline{\Delta}_0$ are

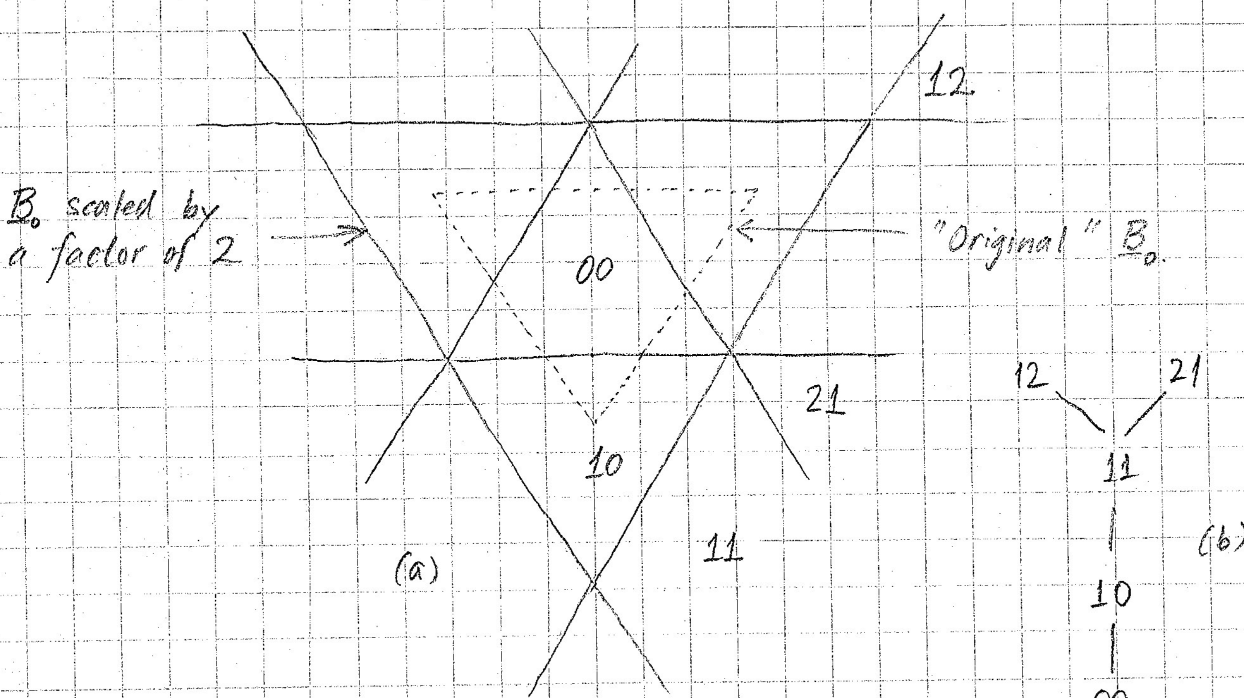


Figure 7

the midpoints of the edges of $\underline{B}_0$, and hence the small 01 cells in Figure 6 have "degenerated" (or "collapsed") to a single point. Hence the cell diagram, given in Figure 7(b), is slightly different from the one in Figure 5(b). Note that 01 is absent from Figure 7(b) not because it is empty, but because it is "degenerate". Cell 20 is also degenerate, and 02 and 22 are empty.

Of course, we may scale B_0 by a factor $\sigma > 2$ (we will use the Greek "sigma" to represent scale factors), with the result as shown in Figure 8.

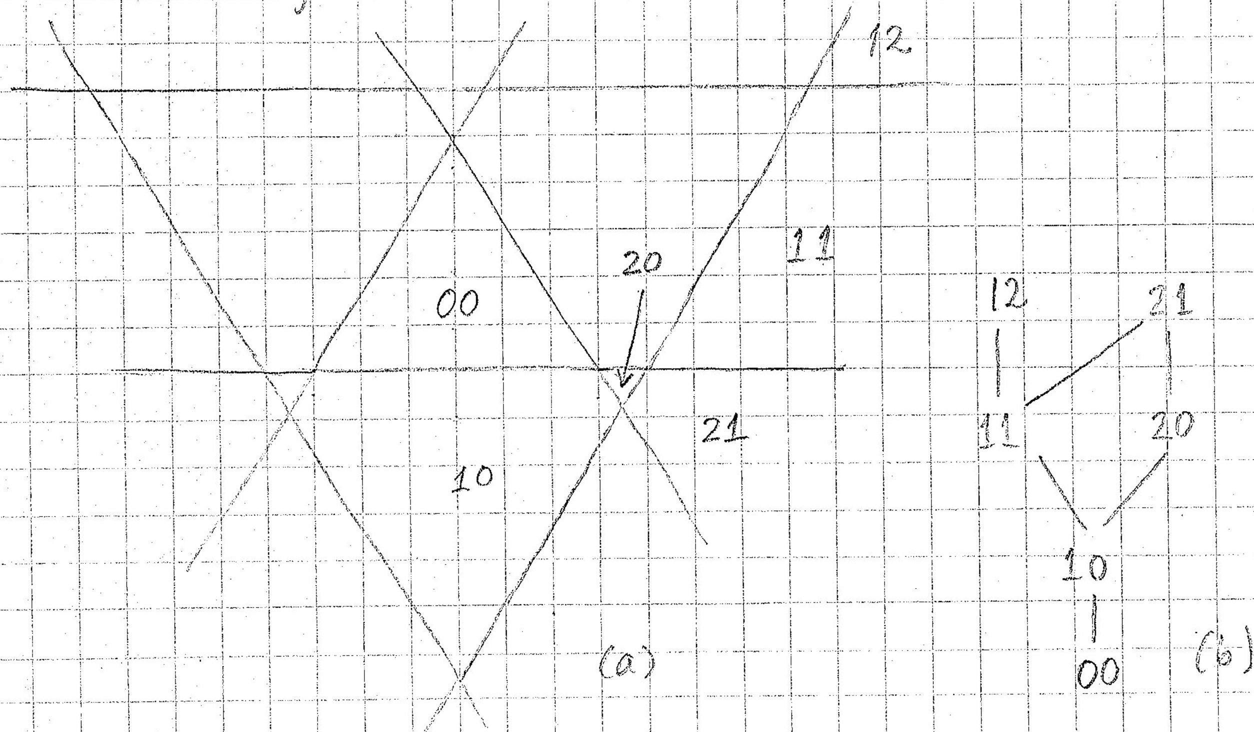


Figure 8

In this case, 01 is now empty (not degenerate, as was the case when $\sigma = 2$), as are 02 and 22. Thus, in this case, we see that any cell which is nonempty is also nondegenerate. Note that this is also the case in Figures 4 and 6. Values of σ yielding such diagrams will be called <u>stable</u>, since for such values of σ, nearby values of σ will yield the same cell diagram. The other values of σ, such as $\sigma = 2$ (see Figure 7) will be called <u>transitional</u>, as they yield configurations which form transitions between stable configurations.

Thus, values of σ near a transitional value result in cell diagrams different than that of the transitional value. Also note that diagrams corresponding to transitional values always contain degenerate regions (i.e., regions which have collapsed to a single point).

Are there transitional values of σ other than $\sigma = 2$ as previously encountered? Some thought yields the discovery of $\sigma = \frac{1}{2}$, shown in Figure 9.

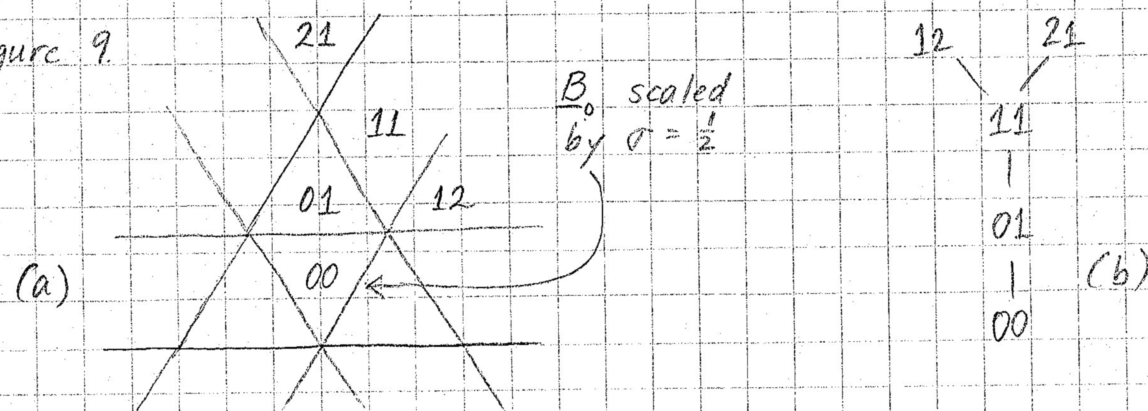


Figure 9

A typical case when $0 < \sigma < \frac{1}{2}$ is shown in Figure 10.

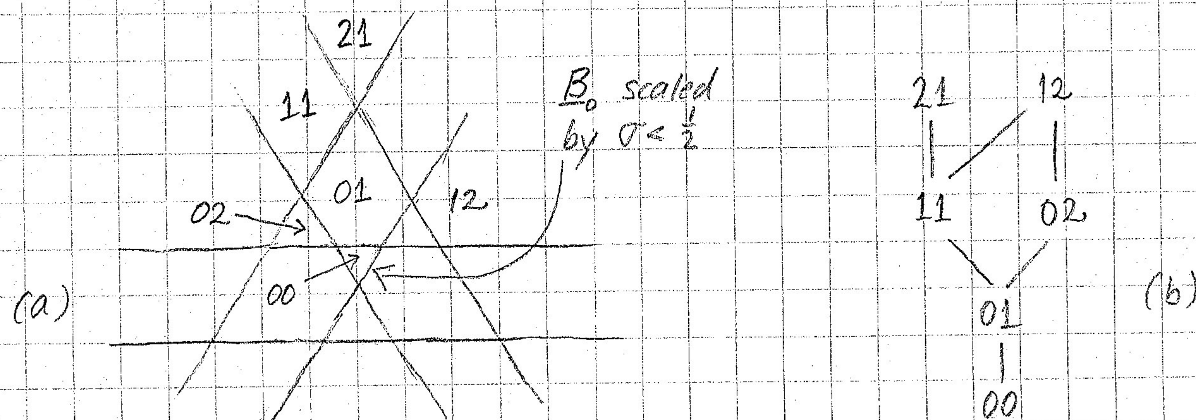


Figure 10

We will summarize our results in a moment; but before doing so, we wish to give an interpretation to the cases $\sigma = 0$ and $\sigma = \infty$. To give meaning to the case $\sigma = 0$, we look closely at Figure 10. As σ nears zero, the triangle $\underline{B}_0$ scaled by σ "shrinks". Were we to draw a diagram for the case, say, $\sigma = \frac{1}{1000}$, we would obtain a diagram such as that in Figure 11(a). $\underline{B}_0$ becomes so small that

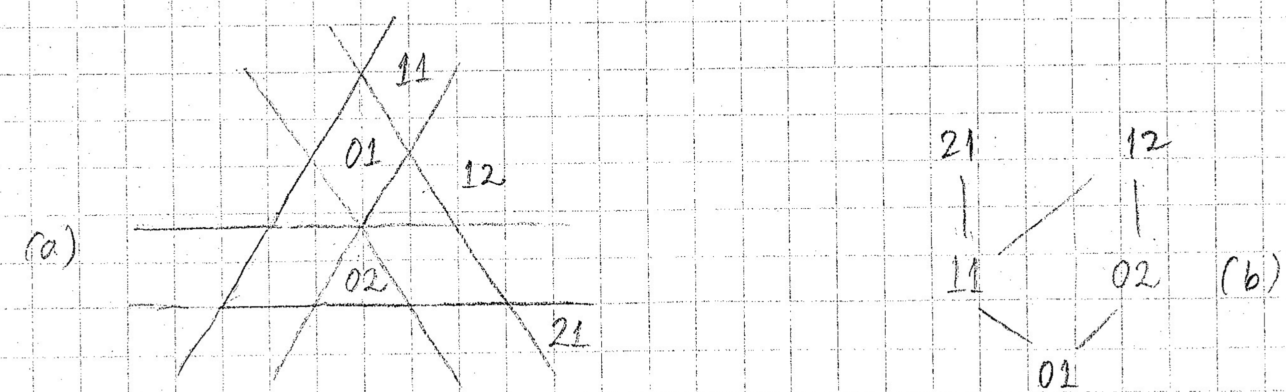


Figure 11

it cannot be distinguished; it appears that the lines containing the edges of $\underline{B}_0$ intersect in a point. Indeed, this is the interpretation assigned to the case $\sigma = 0$: the cell 00 degenerates to a single point, yielding three lines concurrent at the center of $\underline{A}_0$. Thus, the cell diagram in Figure 10(b) becomes that of Figure 11(b) — 00 is not included in Figure 11(b) because it is degenerate. Hence, 0 is a transitional value of σ.

How do we interpret the case $\sigma = \infty$? From the "perspective" of $\underline{A}_0$, as σ increases, $\underline{B}_0$ scaled by σ becomes larger and larger.

But from the "perspective" of $\underline{B}_0$ it appears that $\underline{A}_0$ is shrinking! From $\underline{B}_0$'s perspective, as σ tends to infinity, $\underline{A}_0$ shrinks to a single point. Thus, we interpret the case when $\underline{B}_0$ is scaled by $\sigma = \infty$ to be the case when $\underline{B}_0$ is its original size, but $\underline{A}_0$ is scaled by $\sigma = 0$ (as in the previous discussion). The result is shown in Figure 12. Note that

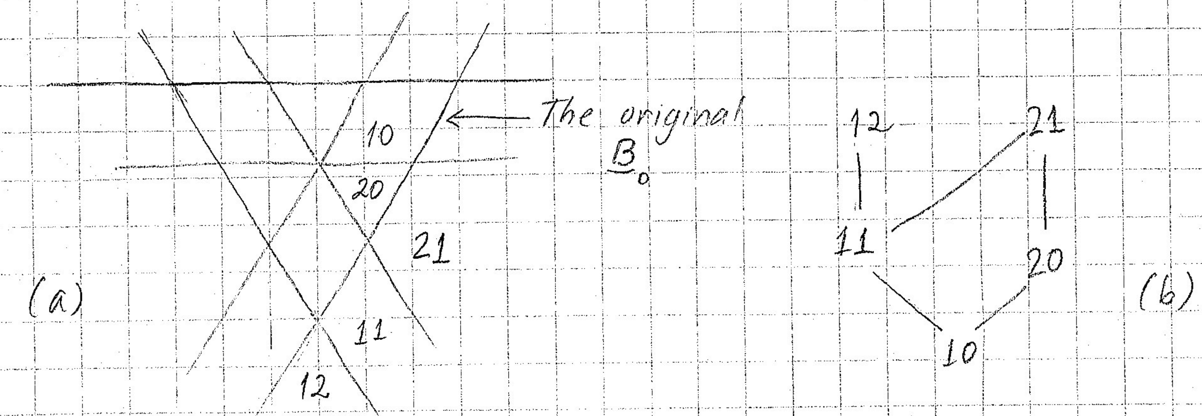


Figure 12

∞ is a transitional value of σ.

The data obtained thus far are summarized in Figure 13. An empty box in the table means that for the value(s) of σ at the top of the column containing the box, the cell at the beginning of the row containing the box is empty. As an example, the cell 02 is empty for $2 < \sigma < \infty$. Similarly, a "•" in a box means that a cell is degenerate, a "B" indicates that a cell is bounded and nondegenerate, and an "∞" means that a cell is unbounded for particular values of σ. The data are obtained from Figure 11 ($\sigma = 0$), Figure 10 ($0 < \sigma < \frac{1}{2}$), Figure 9 ($\sigma = \frac{1}{2}$), Figure 6 ($\frac{1}{2} < \sigma < 2$), Figure 7 ($\sigma = 2$), Figure 8 ($2 < \sigma < \infty$), and Figure 12 ($\sigma = \infty$). Note the symmetry of the entries in

SCALE FACTOR

CELL	$\sigma = 0$	$0 < \sigma < \frac{1}{2}$	$\sigma = \frac{1}{2}$	$\frac{1}{2} < \sigma < 2$	$\sigma = 2$	$2 < \sigma < \infty$	$\sigma = \infty$
00	•	B	B	B	B	B	•
01	B	B	B	B	•		
02	B	B	•				
10			•	B	B	B	B
11	∞	∞	∞	∞	∞	∞	∞
12	∞	∞	∞	∞	∞	∞	∞
20					•	B	B
21	∞	∞	∞	∞	∞	∞	∞
22							

Figure 13

row ij and row ji. Such symmetry will be addressed later.