A Comprehensive Theory of Stellations

What is a stellation of a polyhedron? Miller offers a set of criteria in Coxeter, Du Val, Flather, and Petrie's "The Fifty-Nine Icosahedra". However, Messer notes that these criteria are too broad when considering polyhedra with many faces, as the possibilities become too numerous to adequately discuss. He therefore suggests considering a subclass of stellations which he describes as "fully-supported".

In these notes, I will offer a framework in which a theory of stellations may be discussed. This framework will allow an easy description of the ideas mentioned above, and will also be applicable in two, four, and higher dimensions.

Let a polyhedron $\mathcal{P}$ in n dimensions be given ($n \geq 2$). Let F be an enumeration of the ($(n-1)$-dimensional) faces. For the i^{th} face, let H_i represent the $(n-1)$-dimensional hyperplane including the i^{th} face (line in 2-d, plane in 3-d). Now H_i divides space into two regions; let H_i^0 be that region which includes $\mathcal{P}$, and let H_i^1 be the other. We include H_i in these regions, so that $H_i^0 \cap H_i^1 = H_i$. See Fig. 1 in a 2-d case.

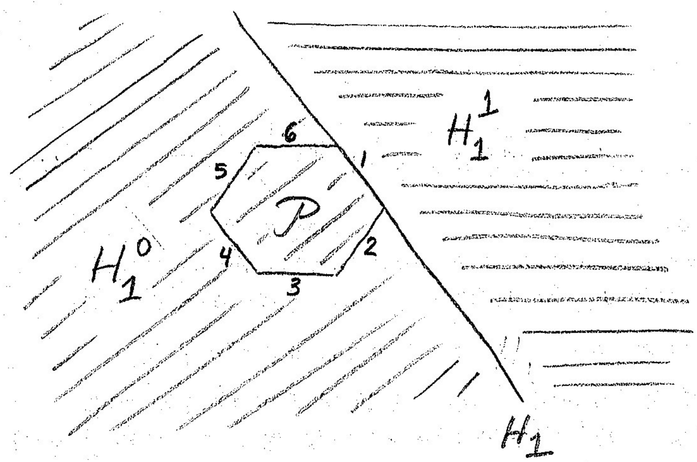


Figure 1

For each face i, let α_i be a binary digit; i.e, $\alpha_i \in \{0,1\}$. Let f be the number of faces of $\mathcal{P}$. Then the intersection

$$\bigcap_{i=1}^{f} H_i^{\alpha_i} = H_1^{\alpha_1} \cap H_2^{\alpha_2} \cap \cdots \cap H_f^{\alpha_f}$$

is either empty, is closed and bounded, or is closed and unbounded (i.e, "infinite"). These are the "cells" into which the facial planes of $\mathcal{P}$ divide space. We denote by $\mathcal{C}$ the set of all nonempty cells, by $\mathcal{C}_b$ the set of bounded cells in $\mathcal{C}$, and by $\mathcal{C}_u$ the set of unbounded cells in $\mathcal{C}$, so that $\mathcal{C} = \mathcal{C}_b \cup \mathcal{C}_u$, $\mathcal{C}_b \cap \mathcal{C}_u = \emptyset$. For brevity, we denote the intersection $H_1^{\alpha_1} \cap \cdots \cap H_f^{\alpha_f}$ by the binary string $\alpha_1 \alpha_2 \cdots \alpha_f$. Figure 2 offers an example in a two-dimensional case.

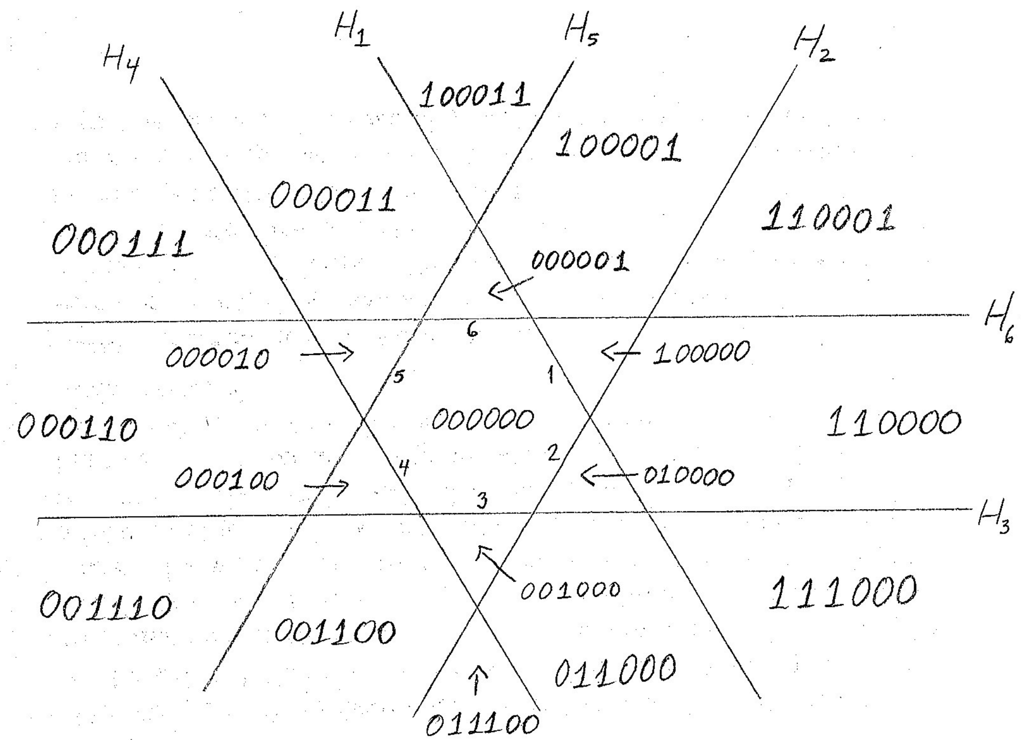


Figure 2.

Note that there are $2^6 = 64$ binary strings of length six, but only 19 regions in Figure 2. But any string of the form 1xx1xx must be empty, since H_1 and H_4 are parallel and hence $H_1^1 \cap H_4^1 = \emptyset$. Likewise, strings of the form x1xx1x and xx1xx1 also represent empty regions. These combinations yield 37 possibilities. The reader is invited to find the 8 strings as yet unaccounted for and see why they yield empty regions.

Adjacency of cells

This method of naming cells gives an easy way to decide when one cell is "on top" of another. For example, consider the cells 100000 and 100001 (see Fig. 2). We see that 100001 is "on top" of 100000, and that these cells are separated by H_6. But note that we oblain the string 100001 from the string 100000 by changing the "0" in the sixth position to a "1". This results in the following general definition.

Definition: Suppose two strings $\alpha = \alpha_1\alpha_2\cdots\alpha_f$ and $\beta = \beta_1\beta_2\cdots\beta_f$ represent cells. We say that "β is on top of α" if the string β is obtained from α by changing a single zero to a one. Thus, if $\alpha_i = 0$ and $\beta_i = 1$, then the cells represented by α and β are separated by H_i. We indicate this relationship by writing "$\alpha \prec \beta$".

Symmetry considerations.

One universally accepted criterion that a polyhedron be a stellation of $\mathcal{P}$ is that it possess the same group of symmetries as does $\mathcal{P}$. We explore this criterion below.

So let $\mathbb{G}$ be the group of symmetries of $\mathcal{P}$.

Since members of $\mathbb{G}$ transform faces (and hence facial planes) of $\mathcal{P}$ into other faces (or facial planes) of $\mathcal{P}$, then members of $\mathbb{G}$ must also transform cells in $\mathcal{C}$ into other cells in $\mathcal{C}$.

Let us return to our 2-d case. For simplicity, we consider $\mathbb{G}$ to be the group of rotational symmetries of the regular hexagon. (We need only consider reflections in cases such as the icosahedron, where some cells come in enantiomorphic ("left-handed" and "right-handed") pairs.) Because of the way we represent cells, we represent members of $\mathbb{G}$ as permutations of faces, so that

$$\mathbb{G} = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix} \right\}.$$

Rewriting these permutations as products of disjoint cycles yields the more succinct

$$\mathbb{G} = \{ (), (1\,2\,3\,4\,5\,6), (1\,3\,5)(2\,4\,6), (1\,4)(2\,5)(3\,6), (5\,3\,1)(6\,4\,2), (6\,5\,4\,3\,2\,1) \}.$$

So given a string α, we denote by $\mathbb{G}\alpha$ the set of all cells "symmetric" to α, the strings representing which are simply the permutions of α which are obtained by applying members of $\mathbb{G}$ to α. Hence,

$$\mathbb{G}(000000) = \{000000\}$$
$$\mathbb{G}(100000) = \{100000, 010000, 001000, 000100, 000010, 000001\}$$
$$\mathbb{G}(110000) = \{110000, 011000, 001100, 000110, 000011, 100001\}$$
$$\mathbb{G}(111000) = \{111000, 011100, 001110, 000111, 100011, 110001\}.$$

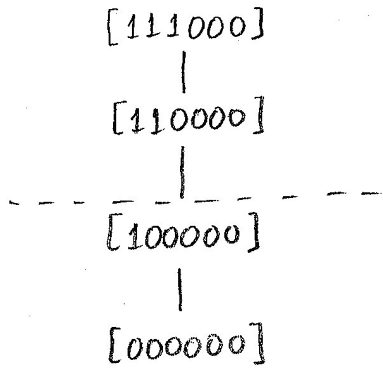
Note that, for example, $\mathbb{G}(100000) = \mathbb{G}(010000)$; as a result, the sets given above exhaust all cells in $\mathcal{C}$. For brevity, we denote $\mathbb{G}(\alpha)$ by $[\alpha]$. We put

$$[\mathcal{C}] := \{ [\alpha] \mid \alpha \in \mathcal{C} \}.$$

Thus, $[\mathcal{C}]$ consists of all cells in $\mathcal{C}$, but "grouped" by symmetry. We may analogously define $[\mathcal{C}_b]$ and $[\mathcal{C}_u]$.

Now we wish to extend the "on top of" relation $\prec$ to $[\mathcal{C}]$. It is clear because of symmetry considerations that if $\sigma \in \mathbb{G}$ is a permutation of the faces of P, and if $\alpha \prec \beta$, then $\sigma\alpha \prec \sigma\beta$. Hence it "makes sense" to write $[\alpha] \prec [\beta]$. Hence, $[100000] \prec [110000]$. But $[100000] = [000001]$, so we must also have $[000001] \prec [110000]$. Thus, we interpret "$[\alpha] \prec [\beta]$" to mean "every cell in $[\beta]$ is on top of <u>some</u> cell in $[\alpha]$".

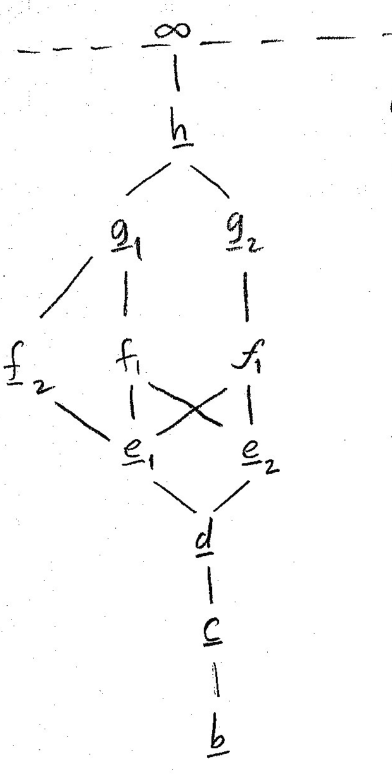
We may form a graphical representation of $\prec$ by depicting the relationship $[\alpha] \prec [\beta]$ as $\begin{array}{c}[\beta]\\ | \\ [\alpha]\end{array}$. Thus, we see that



Remark: Strings with the same number of ones may be said to be at the same "level".

Figure 3.

where the dashed line represents the division be-tween bounded and unbounded cells. Such a diagram is called a "cell diagram". The cell diagram for the icosahedron, with the notations from "The Fifty-Nine Icosahedra", would be



Underlined letters indicate bold-face type.
"∞" represents all unbounded cells.

Figure 4

What are stellations?

It is easy to see that any subset of $[\mathcal{C}]$ represents a set of cells which, when taken together, possess the symmetry of P. But not <u>any</u> subset of $[\mathcal{C}]$ will do. The "connectedness" conditions of Miller ("The 59 Icosahedra", p. 8) impose a further restriction (as noted on p. 15 of the same source). A few definitions are in order first.

Definition: A <u>bounded cell diagram</u> is a cell diagram with only one unbounded node, "∞", which represents all unbounded cells, and is "on top" of the highest cells in the diagram (as in Figure 4).

For simplicity, we are interested in bounded cell diagrams as we are interested in bounded (and therefore constructible with paper) stellations.

Definition: A group of nodes of a cell diagram is said to be <u>connected</u> if one may find a path in the cell diagram from any node in the group to any other traveling <u>only</u> through nodes in the group.

Thus, in Figure 4, $\{\underline{e}_1, \underline{f}_2, g_1\}$ is connected, but $\{\underline{e}_1, \underline{f}_2, g_2\}$ is not, as traveling from $\underline{e}_1$ to g_2 requires traveling through at least one node not in the set $\{\underline{e}_1, \underline{f}_2, g_2\}$.

Now for the definition:

Definition : A subset S of nodes of a bounded cell diagram is said to be a stellation if

1. S is a connected set of nodes not containing "∞",
2. The set of nodes not in S is also connected

Thus, $\{\underline{e}_1, \underline{f}_2, f_1, g_1\}$ in Figure 4 would represent a stellation, whereas $\{\underline{e}_1, f_1, g_1\}$ would not — removing $\{\underline{e}_1, f_1, g_1\}$ "isolates" $\underline{f}_2$ and results in a disconnected group of nodes. Note also that any group of nodes containing $\underline{d}$ must also contain $\underline{c}$ and $\underline{b}$; otherwise removing $\underline{d}$ would "disconnect" $\underline{c}$ and $\underline{b}$ from "∞".

This definition yields precisely the 59 icosahedra. It is useful as it gives a criterion which may easily verified by a computer program. However, the problem is to find an exhaustive list of stellations when the bounded cell diagram gets large.

<u>Definition</u>: A subset S of nodes is said to represent a <u>fully-supported</u> stellation if S represents a stellation and

1. For all $[\alpha] \in [\mathcal{C}]$ and $[\beta] \in S$,

$$[\alpha] \prec [\beta] \Rightarrow [\alpha] \in S.$$

Thus, S is fully-supported if it contains all nodes "below" each node in S.