PERFECT POLYHEDRA AND THEIR DUALS

Concerning the stellations of variations on Archimedean forms, Wenninger remarks on p. 52 of Dual Models, "These various interpenetrations of two basic forms are instances showing how variations of Archimedean forms can often lead to different but closely related shapes, some quite obviously more aesthetically pleasing than others." How pregnant this remark! We take this sentiment to heart in exploring variations on uniform polyhedra.

PART I: Perfect polyhedra with octahedral symmetry

Recall that $\underline{78}$ is a member of $S[\underline{C}, 0\underline{O}]$, $\underline{92}$ is in $S[\underline{C}, \frac{\sqrt{2}-1}{2}\underline{O}]$, the octahedron $\frac{1}{2}\underline{O}$ and the stella octangula are inscribed in $\underline{C}$, the truncated octahedron is in $S[\underline{C}, \frac{3}{4}\underline{O}]$, the cuboctahedron is in $S[\underline{C}, \underline{O}]$, the truncated cube is in $S[\underline{C}, \frac{\sqrt{2}+1}{2}\underline{O}]$, and $\underline{C}$ is inscribed in $\frac{3}{2}\underline{O}$.

Let us write the sequence of scale factors of $\underline{O}$:

$$0, \frac{\sqrt{2}-1}{2}, \frac{1}{2}, \frac{3}{4}, 1, \frac{\sqrt{2}+1}{2}, \frac{3}{2}.$$

Don't a few of the terms look "out of place"? Would it not, somehow, be "perfect" if the "$\frac{\sqrt{2}-1}{2}$" were actually "$\frac{1}{4}$", and the "$\frac{\sqrt{2}+1}{2}$" were "$\frac{5}{4}$"? And what of the series: $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}$? We will investigate the effects on the uniform polyhedra by "forcing" our scale factors to conform to our "perfect sequence."

For reference, we include below a "representative" of the lines which are the intersections of the octahedral faces with a given square face for the various possible scale factors.

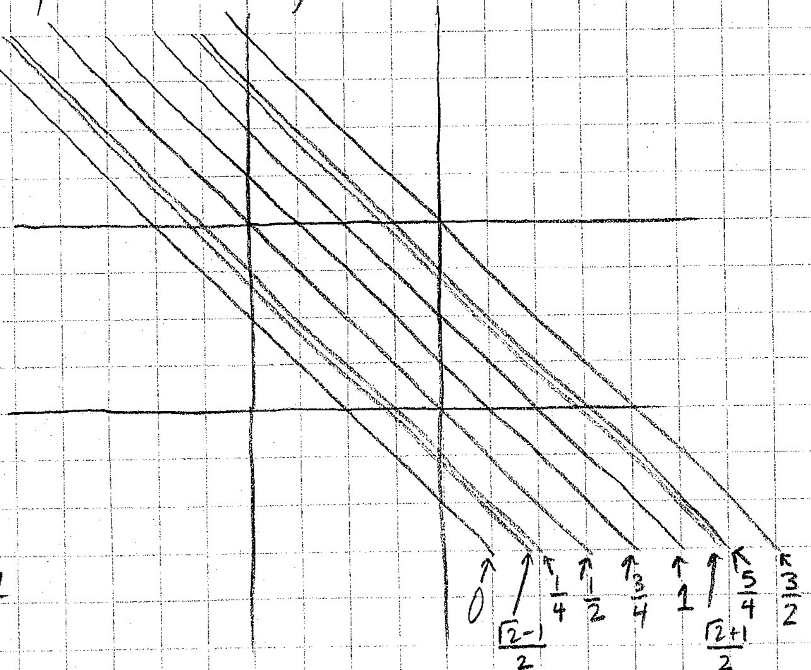


Figure 1

We first consider the truncated cube and include partial stellation diagrams for scale factors $\rho = \frac{\sqrt{2}+1}{2}$ and $\rho = \frac{5}{4}$.

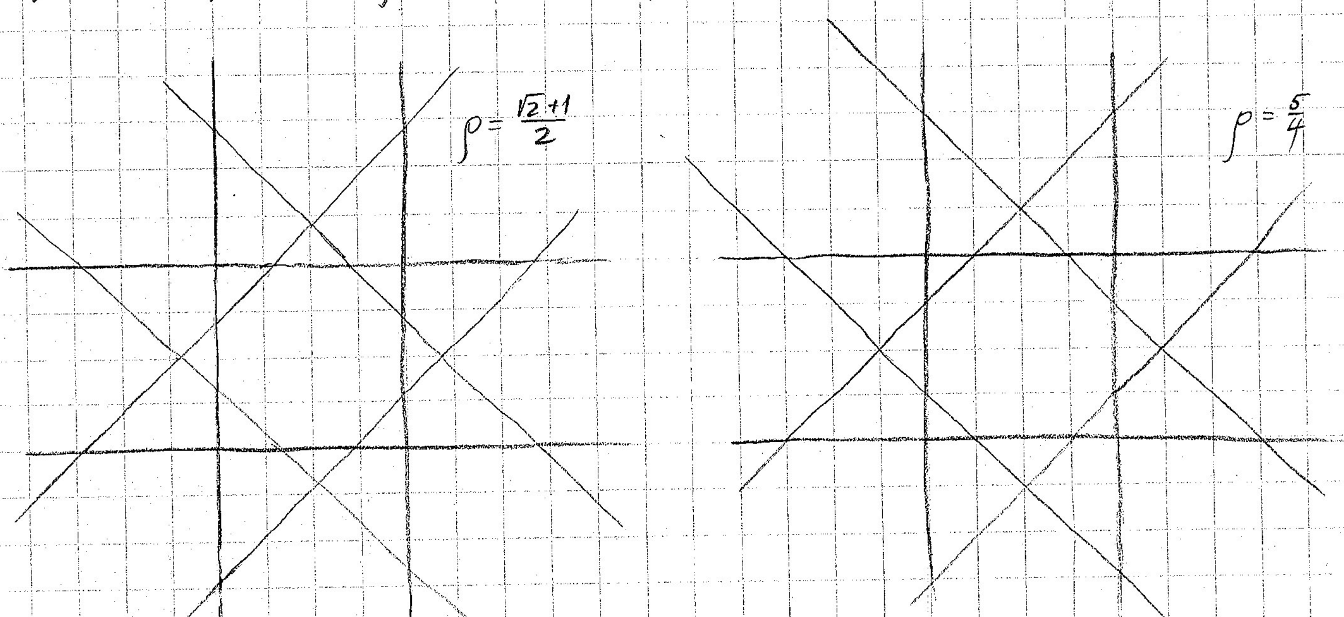
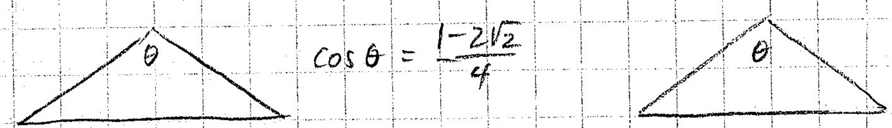


Figure 2

Noteworthy differences between the two cases:

1. Coordinates for $\frac{5}{4}$-truncated cube (and hence its dual) are all rational (those for the $\frac{\sqrt{2}+1}{2}$-truncated cube are not).
2. Rotational symmetries of the octagonal faces of the $\frac{5}{4}$-tr. cube are symmetries of the solid; a $\frac{1}{8}$-rotation of a regular octagonal face of the $\frac{\sqrt{2}+1}{2}$-tr. cube is not a symmetry of the cube.
3. As for the faces of the duals, we have



$\cos\theta = \frac{1-2\sqrt{2}}{4}$

$\sin\theta = ??$

$\frac{\sqrt{2}+1}{2}$-tr. cube

$\cos\theta = -\frac{8}{17}$

$\sin\theta = \frac{15}{17}$

$\frac{5}{4}$-tr. cube

Figure 3

Although the "θ"'s differ by less than a degree, we see a Pythagorean triple in the $\frac{5}{4}$-tr. cube case!

4. The volume of the $\frac{5}{4}$-tr. cube is a rational multiple ~ $\frac{47}{48}$ ~ of that of the cube; the $\frac{\sqrt{2}+1}{2}$-tr. cube is not.

These differences prompt a further investigation. So we call the octagons generated by the case $\rho = \frac{5}{4}$ <u>perfect octagons</u>, and the corresponding inscribed octagrams <u>perfect octagrams</u>. Polyhedra constructed by altering uniform polyhedra by transforming regular octagons (or octagrams) into perfect octagons (or octagrams) are called <u>perfect polyhedra</u>.

Now $\underline{78}$, the cube, the octahedron, the cuboctahedron, and the truncated octahedron are "already" perfect, as none of their faces are octagon/grams. As far as $\underline{92}$ is concerned, it may be shown that the octagrams are perfect when we alter $\underline{92}$ so that it is a member of $S[\underline{C}, \frac{1}{4}\underline{O}]$. The octagrams here are as described in Figure 2. It follows from examining the shorter side of the octagram that the visible parts of the triangular faces are obtained by dividing the sides of the triangles into equal sevenths.

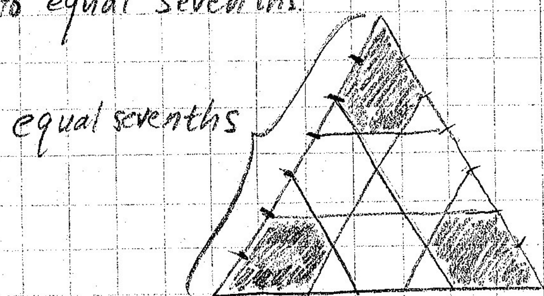


Figure 4: Triangular facial net for perfect version of $\underline{92}$

These adjustments of the truncated cube and $\underline{92}$ give us our "perfect" sequence.

What about the other uniform polyhedra? Well, we need to begin including $\underline{R}$, the rhombic dodecahedron, which is the dual of the cuboctahedron. We consider $\underline{R}$ to the the convex hull of $\underline{C}$ together with $\underline{O}$, so that the vertices of the cuboctahedron are in fact the barycenters of the faces of $\underline{R}$.

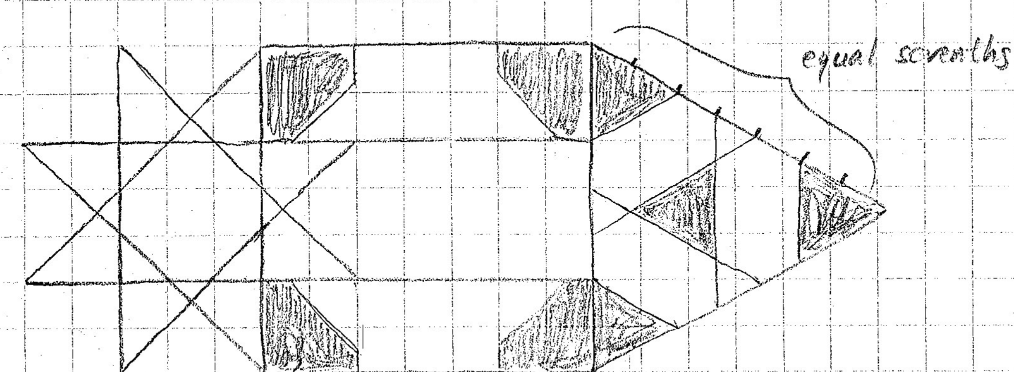
As it happens, if we truncate $\underline{R}$ by creating the polyhedron $t\underline{R}$ whose vertices are the midpoints of the edges of $\underline{R}$, we get the perfect version of the rhombicuboctahedron! This means that there are six squares and eight triangles whose edges are in the ratio $\sqrt{2}:1$, and that the twelve squares of <u>13</u> corresponding to the edges of the cube (or octahedron) are replaced by rectangles whose sides are in the ratio $\sqrt{2}:1$. This results in cross-sectional perfect octagons. Note that the rectangles have only $180°$ rotational symmetry – as is the case for the axis of symmetry through the centers of the rectangles – so that the rectangular faces reflect actual symmetries of the solid (as opposed to the squares, which suggest a fourfold rotational symmetry).

Also note (by duality) that this is the same solid obtained by taking as vertices the midpoints of the edges of the cuboctahedron. One may show that the perfect rhombicuboctahedron belongs to $S[\underline{C}, \underline{O}, \frac{7}{10}\underline{R}]$ (compare this to the rhombicuboctahedron, which is in $S[\underline{C}, (\sqrt{2}-\frac{1}{2})\underline{O}, \frac{1}{\sqrt{2}}\underline{R}]$). Again, a simplification of algebra.

Of course, <u>69</u> is closely related. <u>69</u> belongs to $S[\underline{C}, (\sqrt{2}-\frac{1}{2})\underline{O}, (\sqrt{2}-1)\underline{C}]$. And the perfect version? When the cross-sectional octagons are perfect, we see that the perfect version of <u>69</u> is in $S[\underline{C}, \underline{O}, \frac{1}{2}\underline{C}]$. What could be simpler? By the same token, we see that the perfect version of <u>86</u> belongs to $S[\frac{1}{2}\underline{C}, \frac{7}{10}\underline{R}]$.

But, alas, everything has its price. We see, for example, that the faces of the dual of the perfect rhombicuboctahedron are not quadrilaterals (as is the case with the dual of 13). This results from the fact that we can not inscribe a circle in a perfect octagon so that it touches each side, and hence there is no sphere to which all of the edges of a perfect polyhedron (some of whose faces are perfect octagons/grams) are tangent.

As far as 77 goes, we see that the facial planes of the octagrams and squares are the same as those for 69, and the facial planes of the triangles of 69 are moved slightly towards the center of the solid, giving half of the diagonals of the octagrams. Thus for the perfect version of 77, we simply use the perfect version of 69 as our model, but we move the octahedral facial planes toward the center of the solid, thereby creating perfect octagrams. Thus we see that the perfect version of 77 belongs to $S[\underline{C}, \frac{3}{4}\underline{O}, \frac{1}{2}\underline{C}]$. Visible parts of the facial planes are as below (where the triangle is subdivided as in Fig. 4).



Scale for square corners:

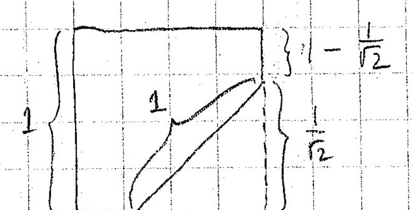


Figure 5

Now let's take a look at a perfect version of 85. Given a cube and its circumscribed rhombic dodecahedron (i.e. that rh. dodeca containing its edges - which we denoted by R), we see that four of the facial planes of R must therefore intersect the facial plane of a cubical face as follows:

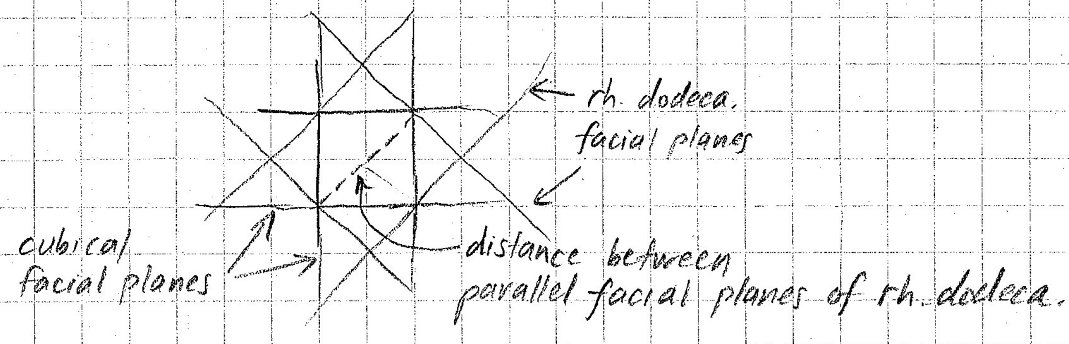


Figure 6

Now the octagrams of the perfect version of 85 must be as the case $\rho = \frac{5}{4}$ of Figure 2. Here, the distance between the "diagonal" lines is precisely $\frac{3}{4}$ the length of the diagonal of the square face (compared with the full length of the diagonal in Figure 6). Thus, twelve of the "squares" in the p.v. (perfect version) of 85 (actually, they are rectangles whose sides are in the ratio $5\sqrt{2} : 7$, very nearly a square) lie in the facial planes of $\frac{3}{4}$R. Of course, the "real" squares and the triangles would be as in the p.v. of 77. Thus we see that the p.v. of 85 is in $S[\underline{C}, \frac{3}{4}\underline{O}, \frac{3}{4}\underline{R}]$.

The diagrams for the visible portions are not difficult to determine in the perfect case. The lines separating portions of the are given in Figure 7, constructed by taking $\frac{e_1}{e_2} = \frac{10}{7\sqrt{2}} = \frac{5\sqrt{2}}{7}$, and by dividing e_1 into ten equal pieces and e_2 into seven equal pieces. (For a look at why "7" and "10", examine closely the sides of the octagram for the case $\rho = \frac{5}{4}$ of Figure 2).

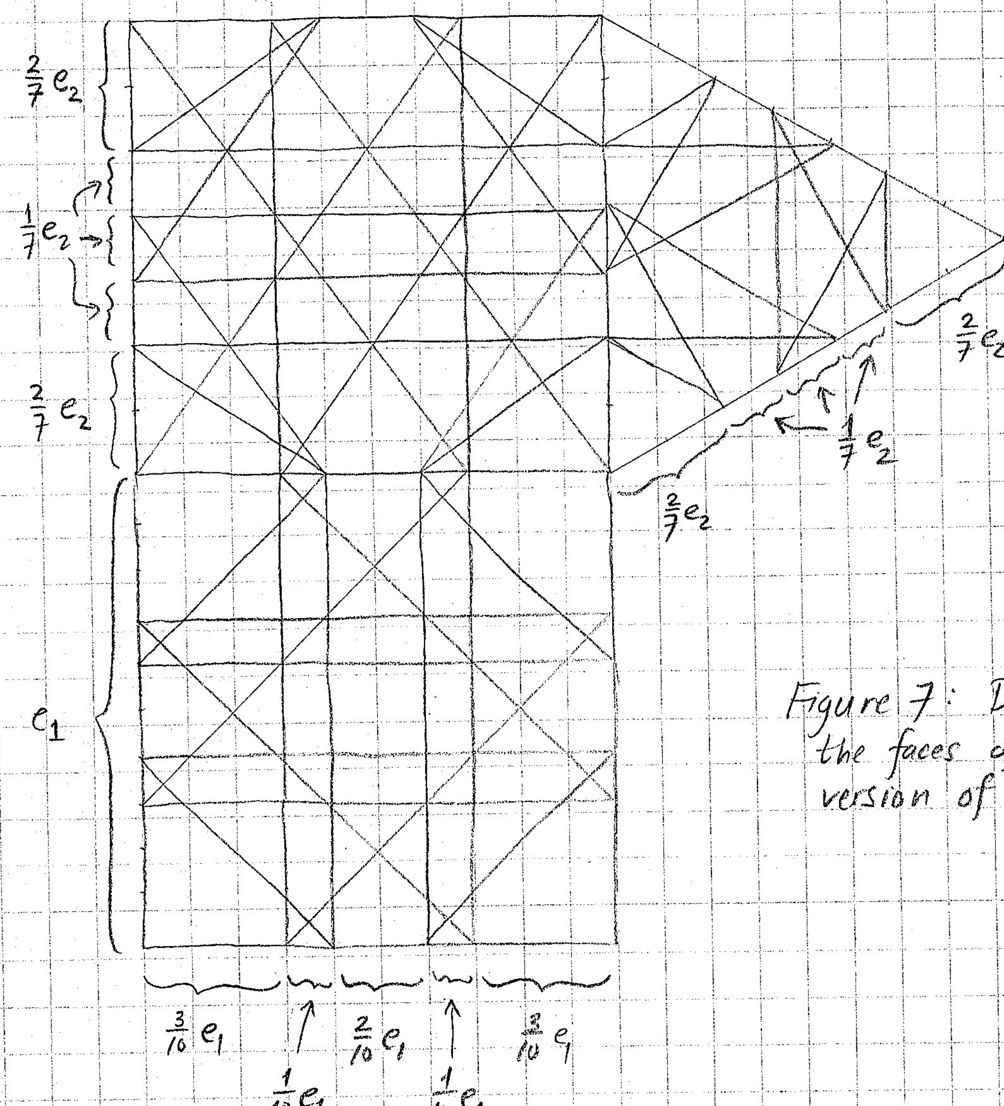


Figure 7: Diagrams for the faces of the perfect version of <u>85</u>.

This analysis makes the alteration of <u>103</u> to a perfect polyhedron a simple task. The p.v. of <u>103</u> belongs to $S[\underline{C}, \frac{3}{4}\underline{R}]$ (See p.v.'s of <u>77</u> and <u>85</u>)

Also, our discussion of "sevenths" and "tenths" makes the facial diagram for the $5\sqrt{2}:7$ rectangles easy to construct from the diagram given on p. 159 of <u>Polyhedron Models</u>.

We now consider a perfect version of <u>79</u>. To do so, we begin by imagining a p.v. of the truncated cube. Next, we move the perfect octagons, keeping them in planes parallel to the faces of the cube, towards the center until the longer edges form the $\#$ sides of a perfect octagram. Now it is clear from Figure 2 that the ratio of $a:b$ for a perfect octagram

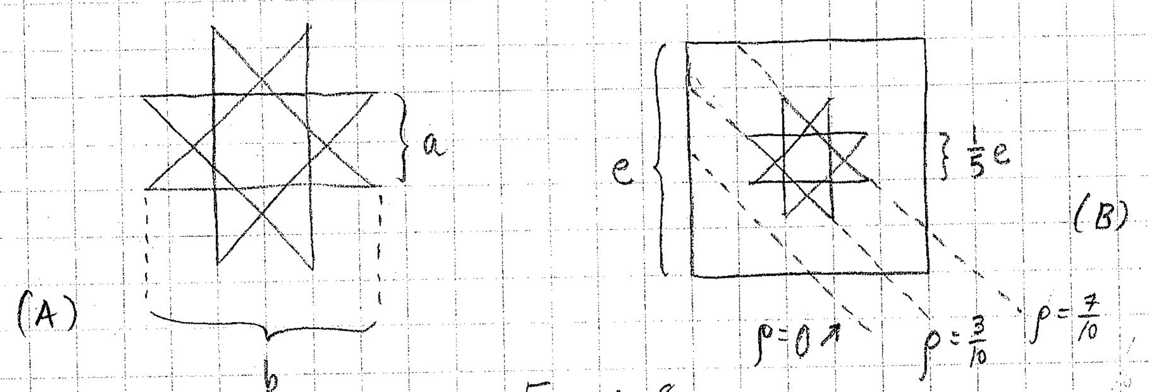


Figure 8

is 2:5 (see Figure 8). Moreover, the longer sides of a perfect octagon are one-half the length of the side of the square in which it is inscribed. Thus, moving the perfect octagons $\frac{2}{5}e$ towards the center achieves the desired result. Thus, Figure 8 (B) reveals that the p.v. of <u>79</u> is in fact a member of $S[\underline{C}, \frac{1}{5}\underline{C}, \frac{7}{10}\underline{O}]$. Although the hexagons are equiangular, they are not regular; the sides alternate in the ratio 5:7.

The facial diagrams are not too difficult to construct as long as we keep in mind that the octagons and hexagons are both adjacent to the octagrams.

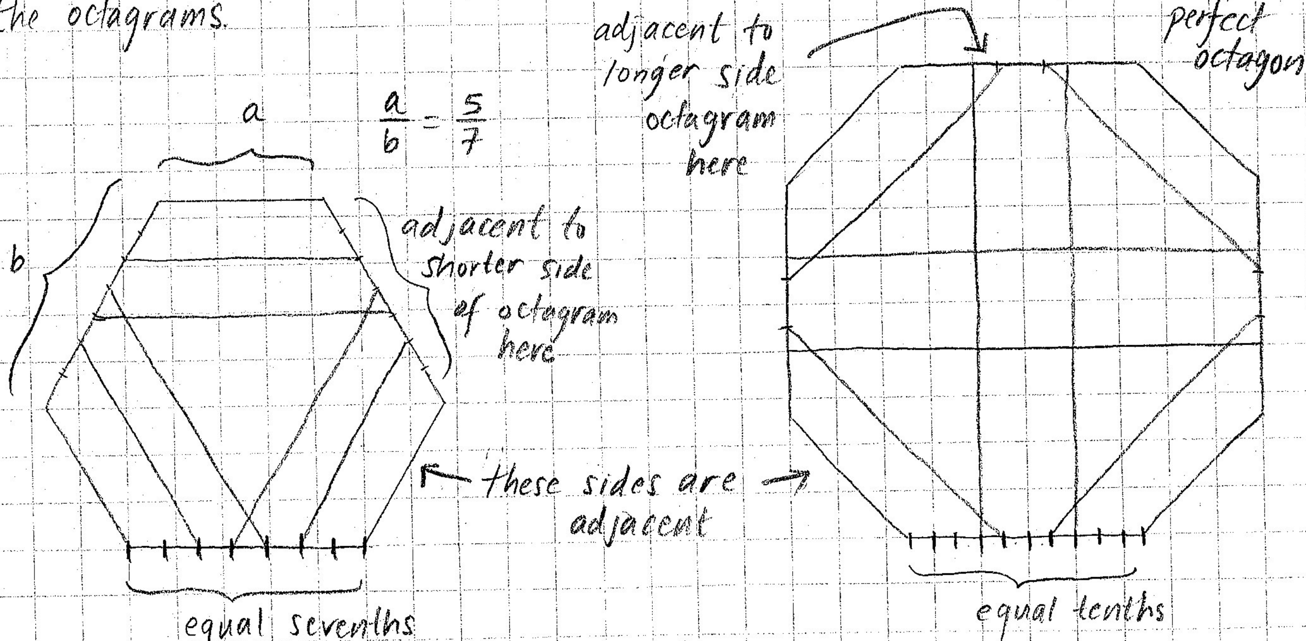


Figure 9

Yet to come:

Perfect versions of 15, 17, and 93.

Part II: Solids with icosahedral symmetry.