Some Irregular Pentagonal Dodecahedra

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1 The α-Icosahedron

It is well-known that an icosahedron or a dodecahedron may be inscribed in a cube so that six of its edges lie within the faces of the cube. In the case that the edge of the circumscribing cube has length 2 (as will be the case throughout this discussion), it may be shown that the length of an inscribed icosahedral edge is $2\tau^{-1}$ (where $\tau = (1+\sqrt{5})/2$), while that of an inscribed dodecahedral edge is $2\tau^{-2}$. (A method for calculating the lengths of these edges is given below.)

In this paper, we wish to consider variations of such inscriptions. Let us begin with the icosahedron. Rather than create a regular icosahedron, we create the twelve vertices of an irregular icosahedron by insisting that its six inscribed edges have length 2α, where $0 \le \alpha \le 1$, rather than $2\tau^{-1}$ (see Figure 1). Using a Cartesian coordinate system based on the circumscribed cube, coordinates for these vertices may be determined (as in Figure 1). Note that there are eight octahedral equilateral triangular faces and twelve isosceles triangular faces, yielding an icosahedron with tetrahedral symmetry (including reflections). We call such an icosahedron an **α-icosahedron.** Note that the case $\alpha = 0$ yields an octahedron (the twelve isosceles triangles degenerate into the edges of the octahedron), while the case $\alpha = 1$ yields a cuboctahedron (the twelve isosceles triangles are in fact right triangles coplanar in pairs, giving the square faces of the cuboctahedron).

The coordinates of any of the isosceles triangles (such as in Figure 2) may be used to calculate the following data. The base and sides of the isosceles triangles are given by

$$b = 2\alpha, \quad a = \sqrt{2(\alpha^2 - \alpha + 1)}, \tag{1}$$

the angles of this triangle are determined by the relationships

$$\cos B = \frac{1-\alpha}{\alpha^2 - \alpha + 1}, \quad \cos A = \frac{\alpha}{\sqrt{2(\alpha^2 - \alpha + 1)}}, \tag{2}$$

and the radius of its circumcircle is

$$r = \frac{\alpha^2 - \alpha + 1}{\sqrt{\alpha^2 - 2\alpha + 2}}. \tag{3}$$

Setting $b = a$ in (1) so that the twelve isosceles triangles are in fact equilateral triangles yields $\alpha = \tau^{-1}$, so that a regular icosahedron of edge length $2\tau^{-1}$ is obtained.

2 The Dual Dodecahedron

We now wish to create the duals of the family of icosahedra just described. We will employ the usual method of polar reciprocity (also known as inversion about a sphere) using a sphere of radius $\rho = 1$. When all of the faces of a polyhedron are regular polygons, it is a simple matter to invert the centers of its faces to obtain the vertices of its dual. For a face which is not a regular polygon, the point to be inverted is that point on the face (or its facial plane) which, when joined to the origin, yields a segment orthogonal to the facial plane of that face. Since the vertices of an α-icosahedron lie on a sphere, the analogous point for an isosceles triangular face is the center of its circumcircle, which is already known (see (3) and Figure 3). Of course, the analogous point for an equilateral triangular face is its center.

Performing the inversion yields the following twenty vertices for the dual dodecahedron: the point $(1/(\alpha+1), 1/(\alpha+1), 1/(\alpha+1))$ along with the other seven vertices obtained by taking all possible changes of sign of its terms (such as $(1/(\alpha+1), -1/(\alpha+1), -1/(\alpha+1))$, along with the point $(1, \alpha-1, 0)$ and the other eleven vertices obtained by taking all possible even permutations and changes of sign of its terms (e.g., $(0, -1, 1-\alpha)$). Shown in Figure 3(a) are the five icosahedral faces surrounding $(1, 0, -\alpha)$ and the corresponding centers of the circumcircles of these faces, while Figure 3(b) shows the inverses of these centers as the vertices of a pentagonal face of the dual dodecahedron. These particular vertices have coordinates

$$\begin{aligned} A' &= (1-\alpha, 0, -1), \\ B' &= \left(\frac{1}{\alpha+1}, -\frac{1}{\alpha+1}, -\frac{1}{\alpha+1}\right), \\ C' &= (1, \alpha-1, 0), \\ D' &= (1, 1-\alpha, 0), \\ E' &= \left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}, -\frac{1}{\alpha+1}\right). \end{aligned} \tag{4}$$

With these coordinates, we find that the lengths of the edges of the pentagon are

$$[C'D'] = 2(1-\alpha), \quad [A'B'] = [B'C'] = [D'E'] = [E'A'] = \frac{\sqrt{\alpha^4+\alpha^2+1}}{\alpha+1}. \tag{5}$$

We may also find that the angles θ, φ, and ψ as depicted in Figure 3(b) are determined by the relationships

$$\cos\theta = \frac{-\alpha^2}{\sqrt{\alpha^4+\alpha^2+1}}, \quad \cos\varphi = \frac{-\alpha}{\alpha^2+\alpha+1}, \quad \cos\psi = 1 - \frac{2}{\alpha^4+\alpha^2+1}. \tag{6}$$

Note that with $\alpha = \tau^{-1}$, we find that all edges have length $2\tau^{-2}$, and the angles are given by

$$\theta = \varphi = \psi = \arccos\left(-\frac{1}{2\tau}\right) = 108°.$$

3 β-Dodecahedra

Recall that this short discourse began with the discussion of variations of a regular icosahedron inscribed in a cube. We now wish to consider the analogous dodecahedral variations. So rather than create a regular dodecahedron, we create twelve of the twenty vertices of an irregular dodecahedron by insisting that the six edges which lie on the faces of a circumscribing cube have length 2β, where $0 \leq \beta \leq 1$, rather than length $2\tau^{-2}$ (see Figure 4). Each inscribed edge then determines two faces, as illustrated in Figure 4; for example, edge CD is incident to two faces that lie in the planes determined by points C, D and A, and points C, D and F. These twelve faces in turn generate the remaining eight vertices of the dodecahedron (such as B, E and G) which, it is apparent, comprise the vertices of a small cube within the interior of the cube circumscribing the dodecahedron. We call such a dodecahedron a **β-dodecahedron.** Note that such dodecahedra possess the symmetry of a regular tetrahedron (including reflections). The case $\beta = 0$ yields a rhombic dodecahedron (edges such as CD degenerate to a point), while the case $\beta = 1$ yields a cube (the pentagonal faces transform into rectangular half-faces of the original cube).

Now coordinates for the twelve vertices (such as A, C and D) inscribed in the "larger" cube are found as they were for the icosahedron. To find coordinates for the eight vertices of the "smaller" cube (mentioned above), we proceed as follows. Symmetry considerations require that the smaller cube is oriented in the same fashion as the larger so that we may imagine the smaller cube as a copy of the larger, although scaled by a factor of σ, where $0 \leq \sigma \leq 1$. Thus, G would have coordinates of the form (σ, σ, σ) since the corresponding corner of the larger cube, H, has coordinates $(1, 1, 1)$. Knowing coordinates for C, D and F, it is a straightforward exercise in analytic geometry to determine that value of σ for which C, D, F and G all lie in the same plane; the result so obtained is

$$\sigma = \frac{1}{2 - \beta}. \tag{7}$$

Note that as β varies between 0 and 1, σ varies between $1/2$ and 1; the reader is invited to give a geometrical argument explaining this phenomenon.

We are now ready to describe the dodecahedral pentagons, typically shown in Figure 5. The vertices of this pentagon are given by

$$
\begin{aligned}
A &= (\beta, 0, -1), \\
B &= \left(\frac{1}{2-\beta}, \frac{-1}{2-\beta}, \frac{-1}{2-\beta}\right), \\
C &= (1, -\beta, 0), \\
D &= (1, \beta, 0), \\
E &= \left(\frac{1}{2-\beta}, \frac{1}{2-\beta}, \frac{-1}{2-\beta}\right).
\end{aligned}
\tag{8}
$$

Upon comparison, it is evident that replacing β with $1-\alpha$ yields precisely the vertices of the pentagon in Figure 3(b) (see (4)). This is not surprising when one considers the relationship between those α-icosahedral edges lying in the faces of the cube and the corresponding dual dodecahedral edges (see Figure 6, which is modelled on data from Figure 3). The six edges that lie in the faces of the cube are sufficient to determine the dual dodecahedron, and the specification of six such edges completely determines a β-dodecahedron. Thus, we see that a $(1-\alpha)$-dodecahedron is dual to an α-icosahedron, or equivalently, a β-dodecahedron is dual to a $(1-\beta)$-icosahedron.

We remark that the edges of an α-icosahedron intersect those of its dual only when $\alpha = \tau^{-1}$; that is, the icosahedron is regular. In general, the only edges of an α-icosahedron which intersect the corresponding edges of its dual $(1-\alpha)$-dodecahedron are those which lie in the faces of the circumscribing cube. (Of course, one can force the other 24 pairs of edges to intersect by modifying the radius of the sphere of inversion, but then the other six pairs fail to intersect.) Thus, in general, there seems to be no straightforward generalization of the rhombic triacontahedron.

4 Stellations of β-Dodecahedra

A number of very interesting polyhedra arise from stellating β-dodecahedra. Not only are many aesthetically appealing, but several are chiral, and the diversity of forms exceeds that of the regular icosahedral stellations. The stellations of β-dodecahedra fall into two classes, with the regular dodecahedron ($\beta = \tau^{-2}$) forming a "boundary". As a result, the parameter ranges $0 < \beta < \tau^{-2}$ and $\tau^{-2} < \beta < 1$ create sets of topologically distinct stellations. We discuss each in turn, and then make some remarks concerning their relationship to the regular dodecahedral stellations. We shall presume on the part of the reader a familiarity with the basic theory of the stellation process. For a good introduction to the stellation process, see Wenninger's *Polyhedron Models.*

4.1 The parameter range $\tau^{-2} < \beta < 1$.

Since the faces of a β-dodecahedron are congruent pentagons, only one stellation diagram is necessary, which is given in Figure 7 (in this case, $\beta = 0.55$). For the reader interested in creating graphical images of such dodecahedra, Cartesian coordinates for the intersections of the lines in this diagram are given in Table 1; note that the core pentagon is the same as the in Figure 5. Due to symmetry considerations, the subdivisions along the segment FS are respectively congruent to those along WN; and the subdivisions along GV are respectively congruent to those along XU. Lengths of these subdivisions may be calculated from the coordinates in Table 1. Upon comparing coordinates for C and D, and also for B and E, it is evident that reflecting through the line containing F, J, A, and V is equivalent to changing the sign of the y-coordinate. Thus, X would have coordinates $(1/\beta, -1/\beta, 1/\beta)$. For convenience, the following abbreviations are used in Table 1:

$$\Delta_1 := (\beta - 1)^3 + 1, \quad \Delta_2 := (\beta - 1)^3 - 1,$$
$$N_1 := \beta^2 - \beta + 1, \quad N_2 := \beta^2 - \beta - 1,$$
$$N_3 := \beta^2 - 3\beta + 1, \quad N_4 := \beta^2 - 3\beta + 3.$$

The cell adjacency diagram is given in Figure 10, and follows the convention of Peter Messer (see [**3**]). Each circle represents a different type of three-dimensional cell in which space is divided by the facial planes of the

β-dodecahedron, where an asterisk denotes a chiral cell. Annotations are interpreted as in the following example: cell F, a chiral cell, rests atop cells B and C, meeting at two-dimensional facets $2a$ and $2c$, respectively (see Figure 7). The exposed facets are $3c$ and $3e$. In the final stellation, cell G covers $3c$, but $3e$ is still exposed; this is indicated by the connection to the "infinite" cell labelled with ∞ (more about the use of this cell later). The number in each circle represents the "level" of the cell in the sense that all the cells at a given level completely cover (to the extent possible; recall that some facets are exposed in the final stellation) those cells at a level one lower. Thus, beginning with the core dodecahedron, we may build succesive "shells" outward at levels 1, 2, 3 and 4. The mainline stellations thereby produced are shown in Figures ???.

$A = (\beta, 0, -1)$,	$L = (N_2/\Delta_2, -N_4/\Delta_2, N_3/\Delta_2)$,
$B = (\sigma, -\sigma, -\sigma)$,	$M = (1, 2-\beta, 0)$,
$C = (1, -\beta, 0)$,	$N = (1, 1/(1-\beta), 0)$,
$D = (1, \beta, 0)$,	$P = (N_1/\Delta_1, -N_2/\Delta_1, N_3/\Delta_1)$,
$E = (\sigma, \sigma, -\sigma)$,	$Q = (-N_3/\Delta_1, N_1/\Delta_1, N_2/\Delta_1)$,
$F = (1/(1-\beta), 0, \beta/(1-\beta)^2)$,	$R = (N_3/\Delta_2, N_2/\Delta_2, N_4/\Delta_2)$,
$G = (1/\beta, 1/\beta, 1/\beta)$,	$S = (0, 1, -1/(1-\beta))$,
$H = (-N_4/\Delta_2, N_3/\Delta_2, N_2/\Delta_2)$,	$T = (0, \beta/(1-\beta)^2, -1/(1-\beta))$,
$J = (2-\beta, 0, 1)$,	$U = (0, (2-\beta)/(1-\beta)^2, -1/(1-\beta))$,
$K = (-N_2/\Delta_1, -N_3/\Delta_1, N_2/\Delta_1)$,	$V = (-1/(1-\beta), 0, (\beta-2)/(1-\beta)^2)$.

Table 1

Using the criteria for being a stellation given in *The Fifty-Nine Icosahedra,* we find that there are 270 stellations of a β-dodecahedron in this parameter range (including the core dodecahedron). This rather surprising abundance is due to the fact that there are three chiral cells B, E and F. We take a moment to explore a few of the intricacies involved in enumerating the stellations.

	$\emptyset$	B	B^1	F	F^1	B E	B^1 E	B E^1	B^1 E^1	B F	B^1 F	B F^1	B^1 F^1
$\emptyset$	FS	P	√	P	√	√	P	P	√	√	P	√	√
A	FS	FS	FS	×	×	FS	P	FS	FS	√	P	√	√
C	√	×	×	√	√	×	×	×	×	×	√	×	√
D	√	√	√	×	×	√	P	√	√	×	P	√	√
G	√	×	×	√	√	×	×	×	×	√	√	P	√
AC	FS	FS	FS	√	√	FS	P	FS	FS	FS	√	FS	FS
AD	×	√	√	×	×	√	P	√	√	×	×	√	√
AG	×	×	×	×	×	×	×	×	×	√	√	√	√
CD	√	×	√	√	√	×	P	×	√	×	√	×	√
CG	×	×	×	√	√	×	×	×	×	×	√	×	√
DG	√	√	√	√	√	√	P	√	√	×	√	√	√
ACD	√	FS	√	√	√	FS	P	FS	√	FS	√	FS	√
ACG	×	×	×	√	√	×	×	×	×	√	√	√	√
ADG	×	√	√	×	×	√	P	√	√	×	×	√	√
CDG	√	×	√	√	√	×	P	×	√	×	√	×	√
$ACDG$	√	√	√	√	√	√	P	√	√	FS	√	√	√

Table 2

First, we distinguish between the two types of chiral cells. Divide B into chiral cells B^1 and B^2, E into E^1 and E^2, and F into F^1 and F^2 such that cells E^1 and F^1 rest atop B^1, and create an enlarged cell diagram as in Figure 9. Then a collection of cells constitutes a stellation if and only if both it and its complement are connected; i.e., there is a path from each cell to every other cell along edges of the graph connecting the selected cells.

	B^1 F^2	B E F	B^1 E F	B E^1 F	B E F^1	B^1 E^1 F	B^1 E F^1	B E^1 F^1	B^1 E^1 F^1	B^1 E F^2	B E^1 F^2	B^1 E^1 F^2
$\emptyset$	×	√	P	√	√	P	P	P	√	×	P	×
A	×	√	P	√	√	P	P	√	√	×	√	×
C	×	×	P	×	×	√	P	×	√	×	×	×
D	×	×	×	×	√	P	P	√	√	×	√	×
G	×	√	P	√	√	√	P	P	√	×	P	×
AC	√	FS	P	FS	FS	√	P	FS	FS	P	FS	√
AD	×	×	×	×	√	×	P	√	√	×	√	×
AG	×	√	P	√	√	√	P	√	√	×	√	×
CD	√	×	×	×	×	√	P	×	√	×	×	√
CG	×	×	P	×	×	√	P	×	√	×	×	×
DG	√	×	×	×	√	√	P	√	√	P	√	√
ACD	√	FS	×	FS	FS	√	P	FS	√	×	FS	√
ACG	√	√	P	√	√	√	P	√	√	P	√	√
ADG	√	×	×	×	√	×	P	√	√	×	√	√
CDG	√	×	×	×	×	√	P	×	√	×	×	√
$ACDG$	√	FS	×	FS	√	√	P	√	√	×	√	√

Table 2 (cont.)

For example, AB^1CG is not a stellation because if we select cells A, B^1, C and G and *only* the edges of our graph whose ends are among these cells, we find that there is no path from G to the other cells (see Figure 10(a)). Analogously, $AB^1DF^1F^2$ is not a stellation because, although itself connected, its induced *complement* (that is, the graph induced by the remaining cells, as in Figure 10(b)) is not, as there is no path from C to any of the remaining cells.

The condition on the connectedness of the complement implies that any stellation including cell A implicitly includes the core dodecahedron, and so the core dodecahedron is omitted from Figure 9. We also note that the reason the cell labelled "∞" is included is so that the definition of a stellation in terms of connectedness is consistent with the definition given in *The Fifty-Nine Icosahedra* (Miller's rule (iv)).

The 270 possible stellations are listed in Table 2, where the following notes apply:

1. The rows are labelled by the various combinations of the symmetric (that is, nonchiral) cells, where the empty set ($\emptyset$) indicates that no symmetric cells are to be included. The columns are labelled by the various combinations of chiral cells, with the following omissions: *a*) reflected images are omitted, so that since B^1E^1 is included, B^2E^2 is not; since BE^1F^2 is included, BE^2F^1 is not; *b*) any combination which includes E *and* F cells without B cells is omitted, since the only way to "connect" E and F cells is through the B cells; thus, columns headed EF, E^1F, EF^1, E^1F^1, and E^1F^2 are not included; *c*) combinations containing the pair B^1E^2 can never be connected, and so are omitted; *d*) Since some B cell is required to connect an E cell to the other cells, the only stellations including E cells *without* B cells are the provisional stellation E and the chiral stellation E^1; as a result, columns headed "E" and "E^1" are absent.

   To determine the status of a potential stellation (see below), just locate the appropriate entry in Table 2. For example, to find the status of AB^1CEF^2, find the intersection of the row labelled "AC" and the column labelled "B^1EF^2".

2. An "×" in the table indicates that the particular combination of cells does not yield a stellation as described above, while a "√" signifies that the combination does yield a stellation. When a stellation is fully supported in the sense that if a cell is present in a combination, then *every* cell beneath it is also present, the notation "FS" is employed. Finally, a "P" is used to indicate that a stellation is "provisional" in the following sense: a combination of cells determines a *provisional* stellation if removing the horizontal dashed lines in Figure 9 disconnects the cells. Thus, E determines a provisional stellation since E^1 and E^2 share no common facet. Miller, in rule (v) for describing stellations in *The Fifty-Nine Icosahedra,* "allow[s] the combination of an enantiomorphous [chiral] pair having no common part" since he remarks that this occurs in only one case. This phenomenon occurs no fewer than 53 times in Table 2 (including the stellation E), and so warrants a separate classification.

3. Table 2 described the possible 270 stellations, of which 213 are chiral. In addition, there are 35 fully supported stellations and 53 provisional stellations. It is not difficult to convince oneself that no stellation can be both fully supported and provisional.

4.2 The parameter range $0 < \beta < \tau^{-2}$.

The stellation diagram for this range is given in Figure 11 (where in fact $\beta = 0.22$) and the corresponding cell diagram is given in Figure 12, where the same conventions apply as before. Coordinates for points in Figure 11 are still as in Table 1 since they can be obtained as intersections of groups of planes with the same equations as those for the parameter range $\tau^{-2} < \beta < 1$; the parameter β takes on different values here, giving a different topology (that is, connectivity of cells).

The enumeration of the stellations in Table 3 follows the same conventions as before, with the following adjustments. Referring to the notes in **4.1**, we see that (1a) and (1d) still apply (where F is replaced by G), there is no analogue to (1b), and as far as (1c) is concerned, B^1E^2 and $B^1E^2G^1$ are the only inadmissible combinations containing B^1E^2 (the cell G^2 is required to join E^2 to other cells). We also need not consider a column headed E^1G^2, since some B cell is required to connect E^1 to other cells. In addition, because the inclusion of cells A and D *without* C disconnects C from the rest of the cells, we may eliminate rows AD and ADF from consideration. Similarly, since there is no way to connect cells C and F without either A or D, we may elimnate row CF. Finally, since C is not connected to any of the B, E, or G cells, the only stellation in a row labelled C would be C itself (under the "$\emptyset$" column); with this in mind, we do not include a row headed C.

The enumeration results in 205 stellations, 163 of which are chiral. There are 27 fully supported stellations and 51 provisional stellations.

	$\emptyset$	B	B^1	G	G^1	B E	B^1 E	B E^1	B^1 E^1	B G	B^1 G	B G^1
$\emptyset$	FS	P	$\surd$	P	$\surd$	P	P	P	$\surd$	$\times$	$\times$	$\times$
A	$\surd$	FS	FS	$\times$	$\times$	FS	P	FS	FS	$\times$	$\times$	$\times$
D	$\surd$	$\surd$	$\surd$	$\times$	$\times$	$\times$	P	$\times$	$\surd$	$\times$	$\times$	$\times$
F	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	P	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
AC	FS	FS	FS	$\times$	$\times$	FS	P	FS	FS	$\times$	$\times$	$\times$
AF	$\times$	FS	$\surd$	$\times$	$\times$	FS	P	FS	$\surd$	$\surd$	$\surd$	$\surd$
CD	$\surd$	$\times$	$\surd$	$\times$	$\times$	$\times$	P	$\times$	$\surd$	$\times$	$\times$	$\times$
DF	$\times$	$\times$	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	$\times$	$\surd$	$\times$
ACD	$\surd$	FS	$\surd$	$\times$	$\times$	FS	P	FS	$\surd$	$\times$	$\times$	$\times$
ACF	$\times$	FS	$\surd$	$\times$	$\times$	FS	P	FS	$\surd$	$\surd$	$\surd$	$\surd$
CDF	$\times$	$\times$	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	$\times$	$\surd$	$\times$
$ACDF$	$\times$	FS	$\surd$	$\times$	$\times$	FS	$\times$	FS	$\surd$	$\surd$	$\surd$	$\surd$

Table 3

	B^1 G^1	B^1 G^2	E G	E^1 G	E G^1	E^1 G^1	B E G	B^1 E G	B E^1 G	B E G^1	B^1 E^1 G	B^1 E G^1
$\emptyset$	$\times$	$\times$	P	P	P	$\surd$	P	P	P	P	P	P
A	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	P	P	$\surd$	P	P
D	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	P	$\times$	$\times$	P	P
F	$\surd$	$\surd$	$\surd$	$\surd$	P	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	P
AC	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	P	P	$\surd$	P	P
AF	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\times$	FS	$\surd$	$\surd$	$\surd$	$\surd$	P
CD	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	P	$\times$	$\times$	P	P
DF	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	$\times$
ACD	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	P	P	$\surd$	P	P
ACF	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\times$	FS	$\surd$	$\surd$	$\surd$	$\surd$	P
CDF	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\surd$	$\times$
$ACDF$	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\times$	FS	$\times$	$\surd$	$\surd$	$\surd$	$\times$

Table 3 (cont.)

	B E^1 G^1	B^1 E^1 G^1	B^1 E^2 G	B^1 E G^2	B^1 E^2 G^2	B^1 E^1 G^2	B E^1 G^2
$\emptyset$	P	√	×	P	×	×	×
A	√	√	×	P	×	×	×
D	×	√	×	P	×	×	×
F	√	√	√	√	√	√	√
AC	√	√	×	P	×	×	×
AF	√	√	√	√	√	√	√
CD	×	√	×	P	×	×	×
DF	×	√	×	×	×	√	×
ACD	√	√	×	P	×	×	×
ACF	√	√	√	√	√	√	√
CDF	×	√	×	×	×	√	×
$ACDF$	√	√	×	×	×	√	√

Table 3 (cont.)

5 The Regular Dodecahedron

Recall that the regular dodecahedron, in part due to its highly symmetric nature, is usually considered to have just three stellations (other than the core dodecahedron): the small stellated dodecahedron, the great dodecahedron, and the great stellated dodecahedron. Let us call the outermost cells which constitute these stellations X, Y, and Z, respectively, so that X consists of twelve pentagonal pyramids, Y consists of thirty tetrahedral wedges, and Z consists of twenty triangular bipyramids.

That there are just three types of cells is a result of the icosahedral (or dodecahedral) symmetry of the dodecahedron. In this section, we consider the regular dodecahedron as a polyhedron with *tetrahedral* symmetry, and examine its stellations in this context. In other words, we seek groups of dodecahedral cells which possess no more than tetrahedral symmetry.

For example, consider the final stellation of the β-dodecahedron described in §**4.1**. Twenty "spikes" are evident – the twelve D cells, along with the eight E cells (four each of E^1 and E^2) – and as we let the parameter β approach τ^{-2}, we see that these cells "become" the twenty spikes of the great stellated dodecahedron. Thus, from a "tetrahedral" perspective, we can imagine the Z cells as being comprised of D, E^1, and E^2 cells. In a similar fashion, we can imagine the Y cells of the great dodecahedron to be composed of B^1, B^2, and C cells. Finally, the X cells of the small stellated dodecahedron correspond to the A cells. For practical purposes, we ignore the F and G cells as they "disappear" as β approaches τ^{-2}.

Thus, we may consider the cells of the regular dodecahedron as being comprised of the A, B, C, D, and E cells of Figure 9, as shown in Figure 13. Here, however, we link the cells B^1, B^2, and C since they together form the Y cells. Similarly, we link the E^1, E^2, and D cells. Using the conventions of Tables 2 and 3, we enumerate the 41 "tetrahedral" stellations of the regular dodecahedron in Table 4.

One subtlety arises, however, as illustrated in the following example. Consider the set of cells AB^1D. Clearly, A, B^1, and D are connected. But the connectedness of the complement – that is, the cells B^2, C, E^1, E^2, and ∞ – depends of the dashed horizontal link between B^2 and C. Since the C cells are in actuality sandwiched between the A and D cells, the "holes" (i.e., absence of C cells) are not apparent in AB^1D. Hence, from an exterior point of view, the set of cells AB^1D is indistinguishable from AB^1CD. Thus, we disallow AB^1D as a stellation, and in general do not consider horizontal links when determining the connectedness of the complement of a particular set of cells.

	$\emptyset$	B	B^1	E	E^1	B E	B^1 E	B E^1	B^1 E^1
$\emptyset$	FS	P	$\surd$	P	$\surd$	P	P	P	$\surd$
A	FS	FS	FS	$\times$	$\times$	FS	P	FS	FS
C	$\surd$	$\times$	P	$\times$	$\times$	$\times$	P	$\times$	P
D	$\surd$	$\times$	$\surd$	$\times$	P	$\times$	$\times$	$\times$	$\surd$
AC	FS	FS	FS	$\times$	$\times$	FS	P	FS	FS
AD	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
CD	$\surd$	$\times$	$\surd$	$\times$	P	$\times$	$\times$	$\times$	$\surd$
ACD	$\surd$	FS	$\surd$	$\times$	$\times$	FS	$\times$	FS	$\surd$

Table 4

6 References

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2. Coxeter, H. S. M., DuVal, P., Flather, H. T., and Petrie, J. P., *The Fifty-Nine Icosahedra,* University of Toronto Studies, Math. Series No. 6, University of Toronto Press, 1938.

3. Messer, P. W., Stellations of the Rhombic Triacontahedron and Beyond, *Structural Topology,* No. 21, 1995, pp. 25-46.

4. Olshevsky, George, *The 36 Dodecahedra,* private communication.

5. Wenninger, Magnus J., *Polyhedron Models,* Cambridge University Press, 1971.

7 Acknowledgements

Thanks to Vladimir Bulatov for creating graphical images of many of the dodecahedra mentioned in this paper.

Thanks also to George Olshevsky for his enumeration of the tetrahedral stellations of the regular dodecahedron. Our difference in numbers is due to our respective interpretations of what I called provisional stellations.

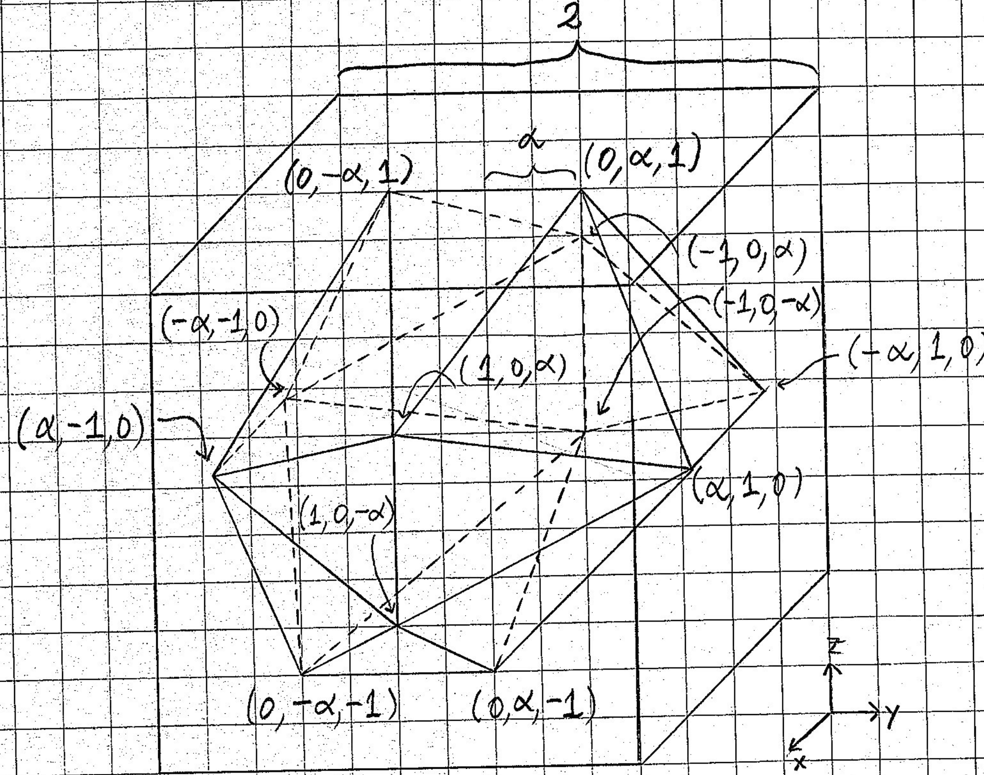


FIGURE 1

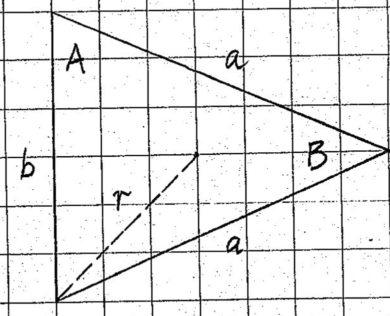


FIGURE 2

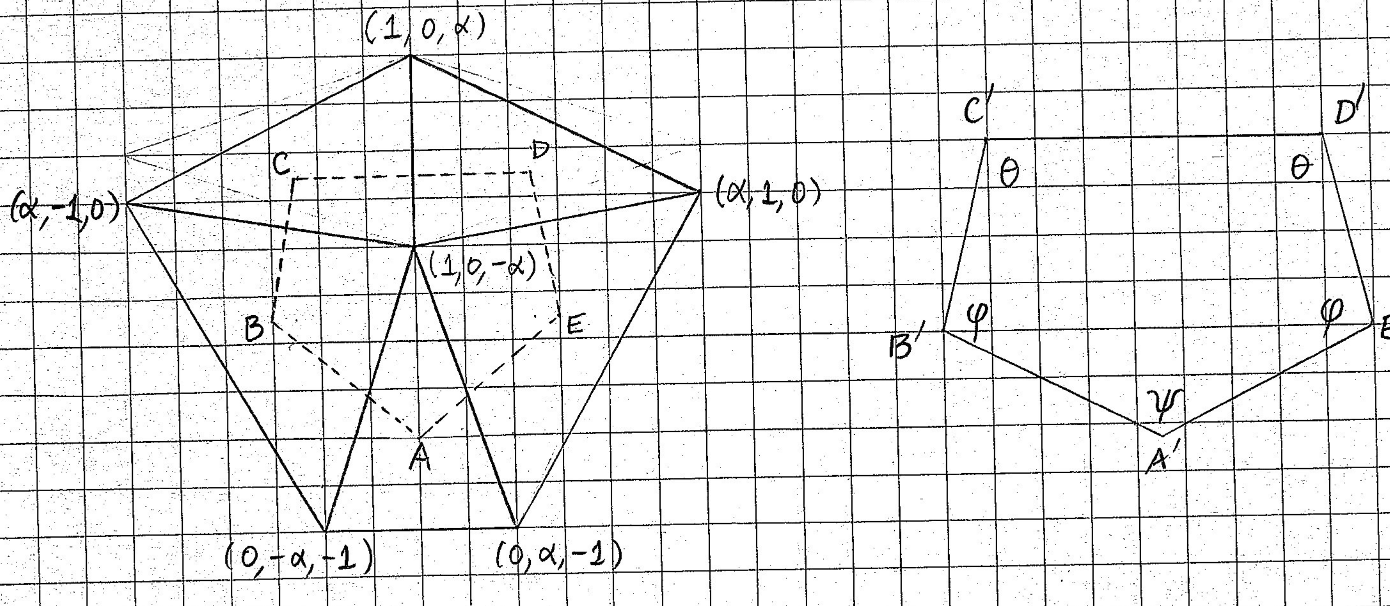


FIGURE 3

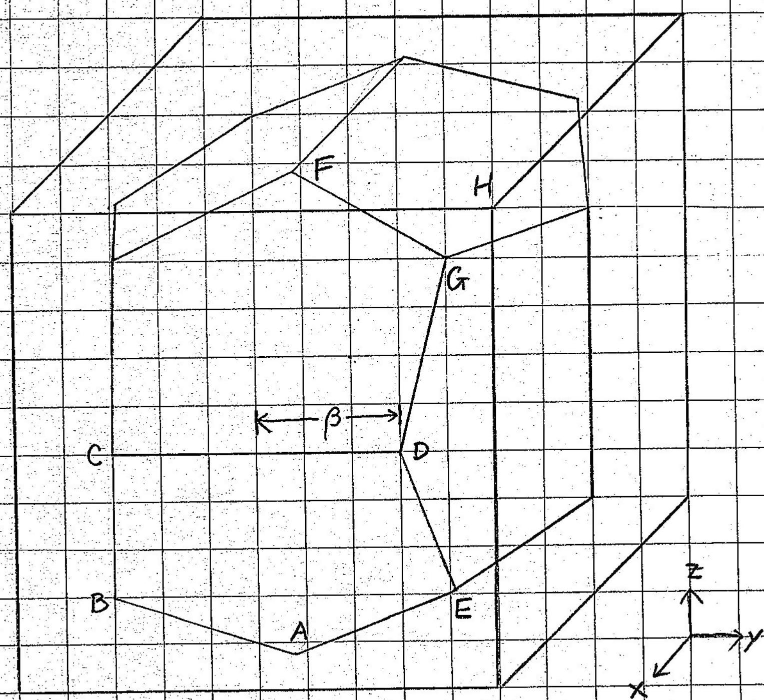
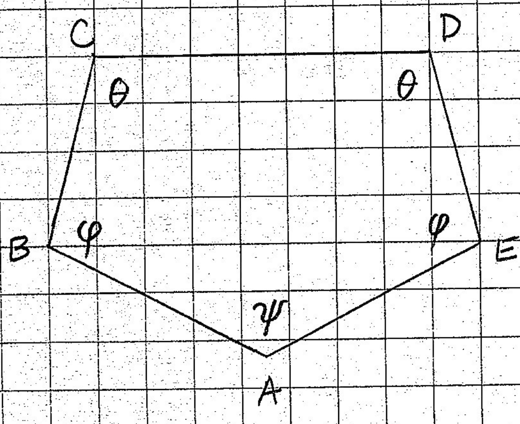


FIGURE 4



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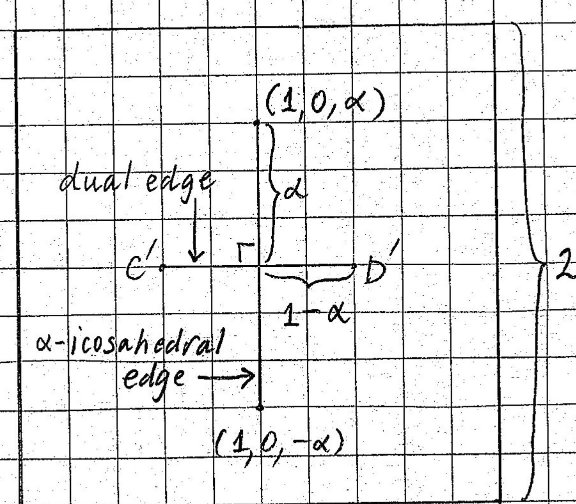

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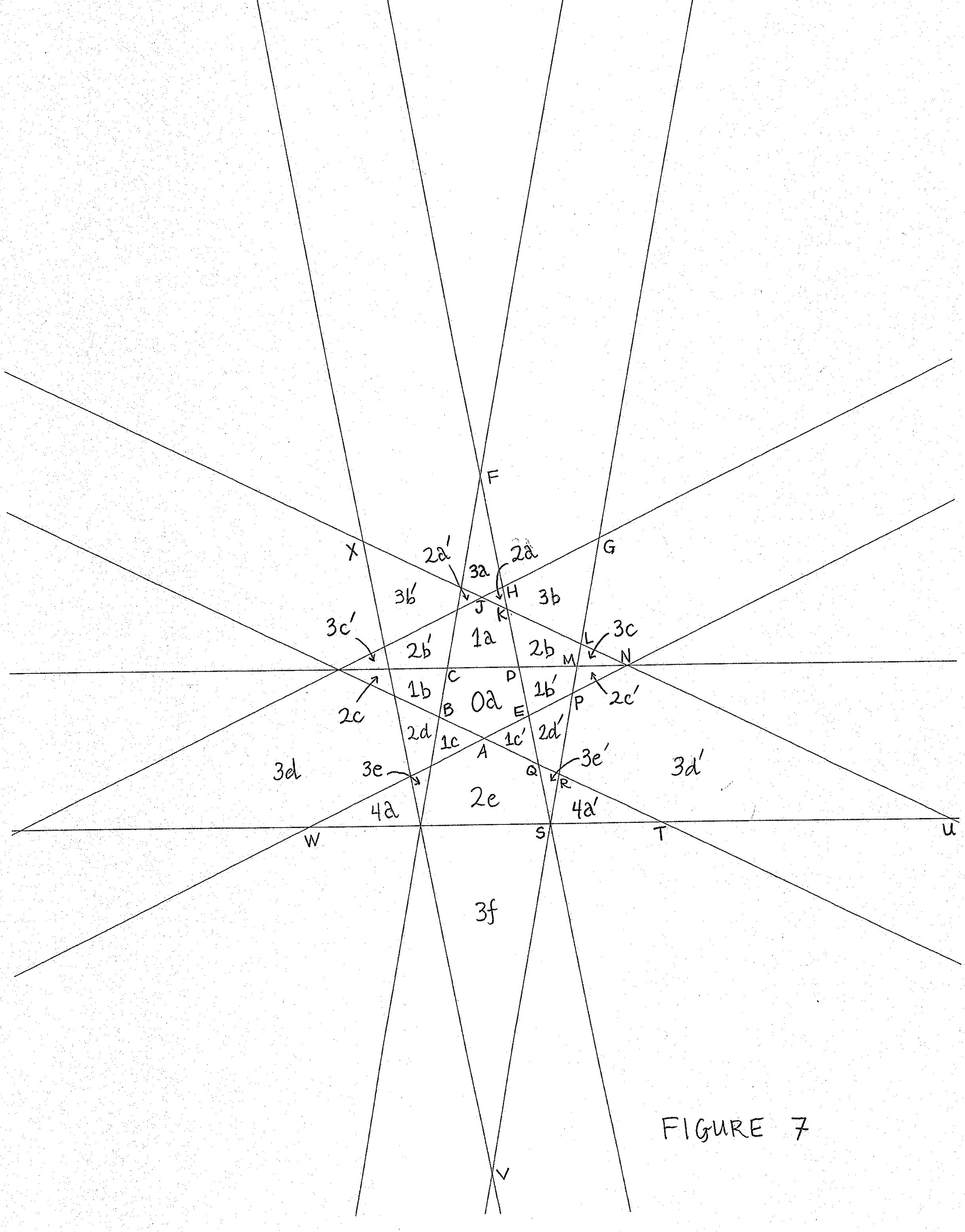
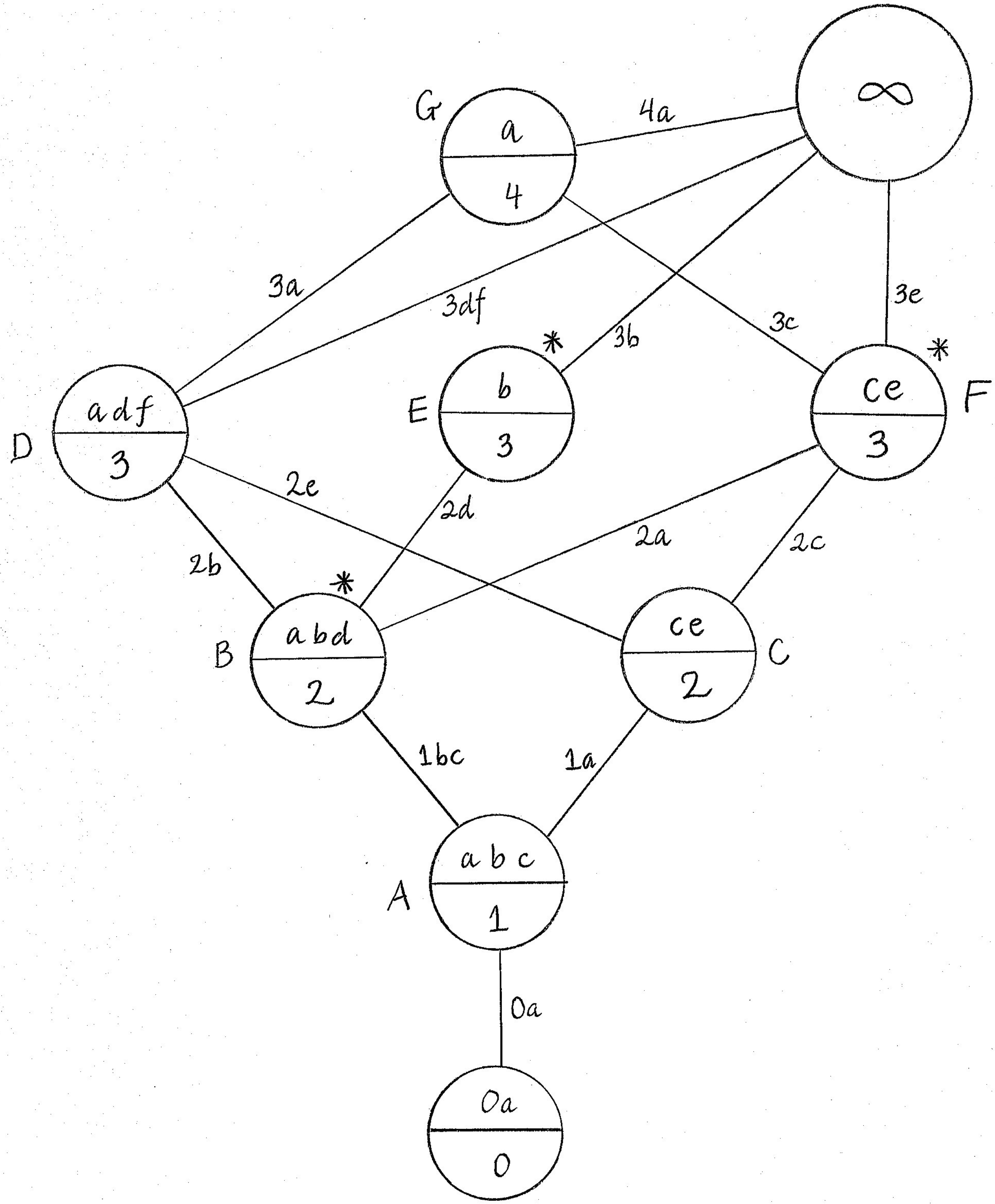


F
X
G
2a′
2a
3a
H
3b′
3b
J
K
3c′
1a
L
3c
2b′
2b
M
N
C
D
1b
0a
1b′
2c
B
E
P
2c′
2d
2d′
1c
1c′
A
3e′
3d
3e
Q
R
3d′
2e
4a
4a′
W
S
T
U
3f
V

FIGURE 7



G
a
4
4a
∞
3a
3df
3b
3c
3e
*
*
D
adf
3
E
b
3
ce
3
F
2e
2d
2a
2c
2b
*
B
abd
2
ce
2
C
1bc
1a
A
abc
1
0a
0a
0

FIGURE 8

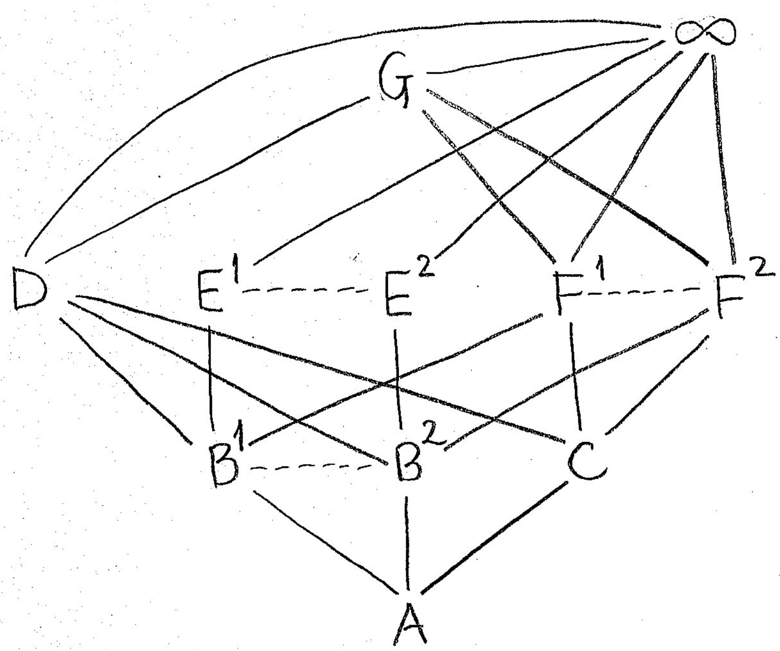


FIGURE 9

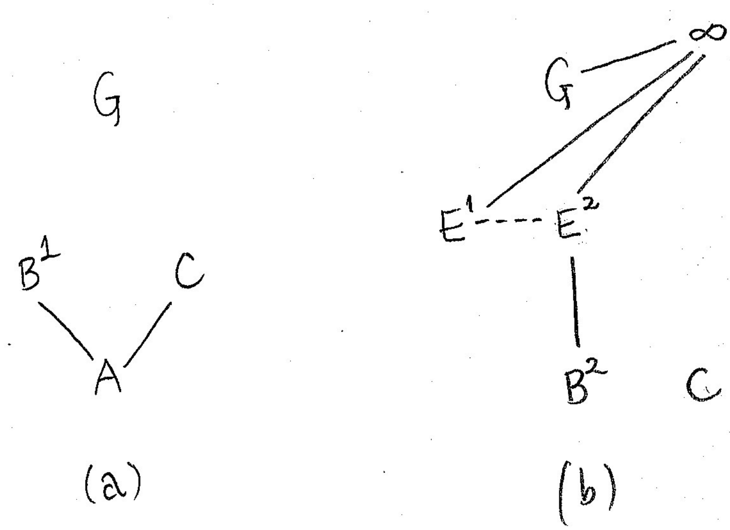


FIGURE 10

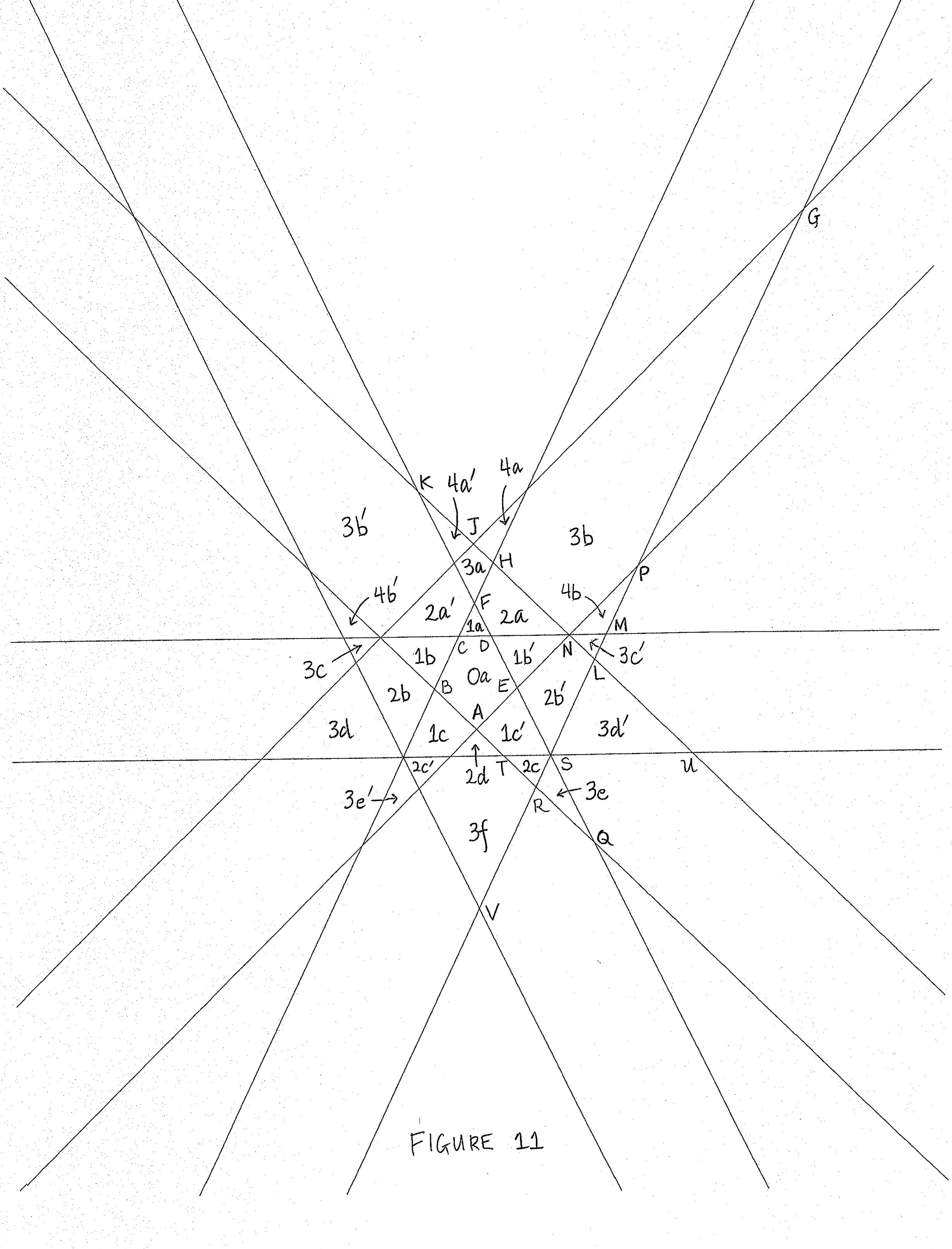

G
K
4a′
4a
J
3b′
3b
3a
H
P
4b′
F
4b
2a′
2a
M
1a
C
D
1b
1b′
N
3c′
3c
Oa
B
E
L
2b
2b′
A
3d
1c
1c′
3d′
2c′
T
2c
S
u
2d
3e′
3e
R
3f
Q
V

FIGURE 11

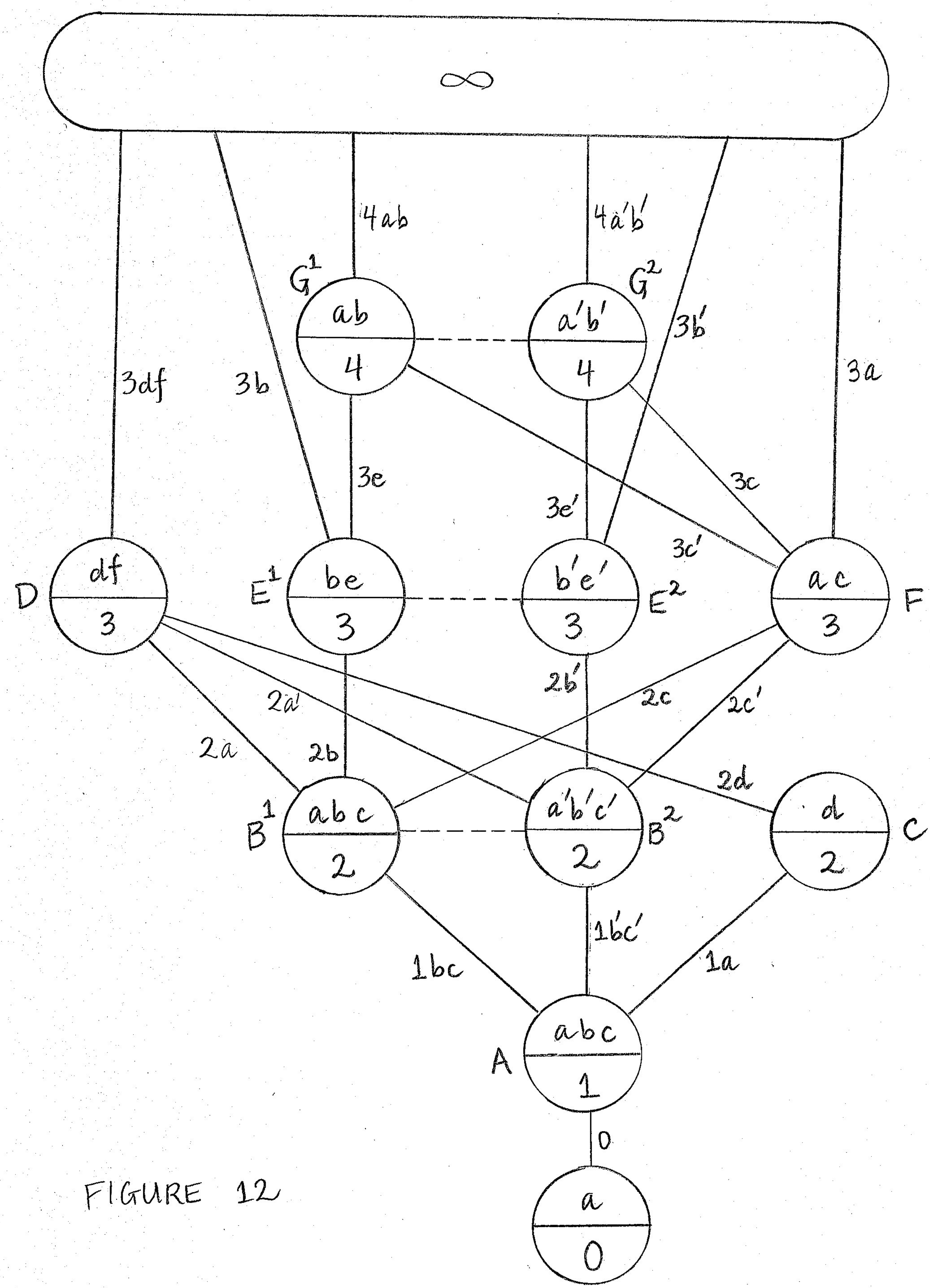

∞
4ab
4a'b'
G1
ab
4
G2
a'b'
4
3b'
3df
3b
3a
3e
3e'
3c
3c'
D
df
3
E1
be
3
b'e'
3
E2
ac
3
F
2b'
2a'
2c
2c'
2a
2b
2d
B1
abc
2
a'b'c'
2
B2
d
2
C
1b'c'
1bc
1a
A
abc
1
0
a
0

FIGURE 12

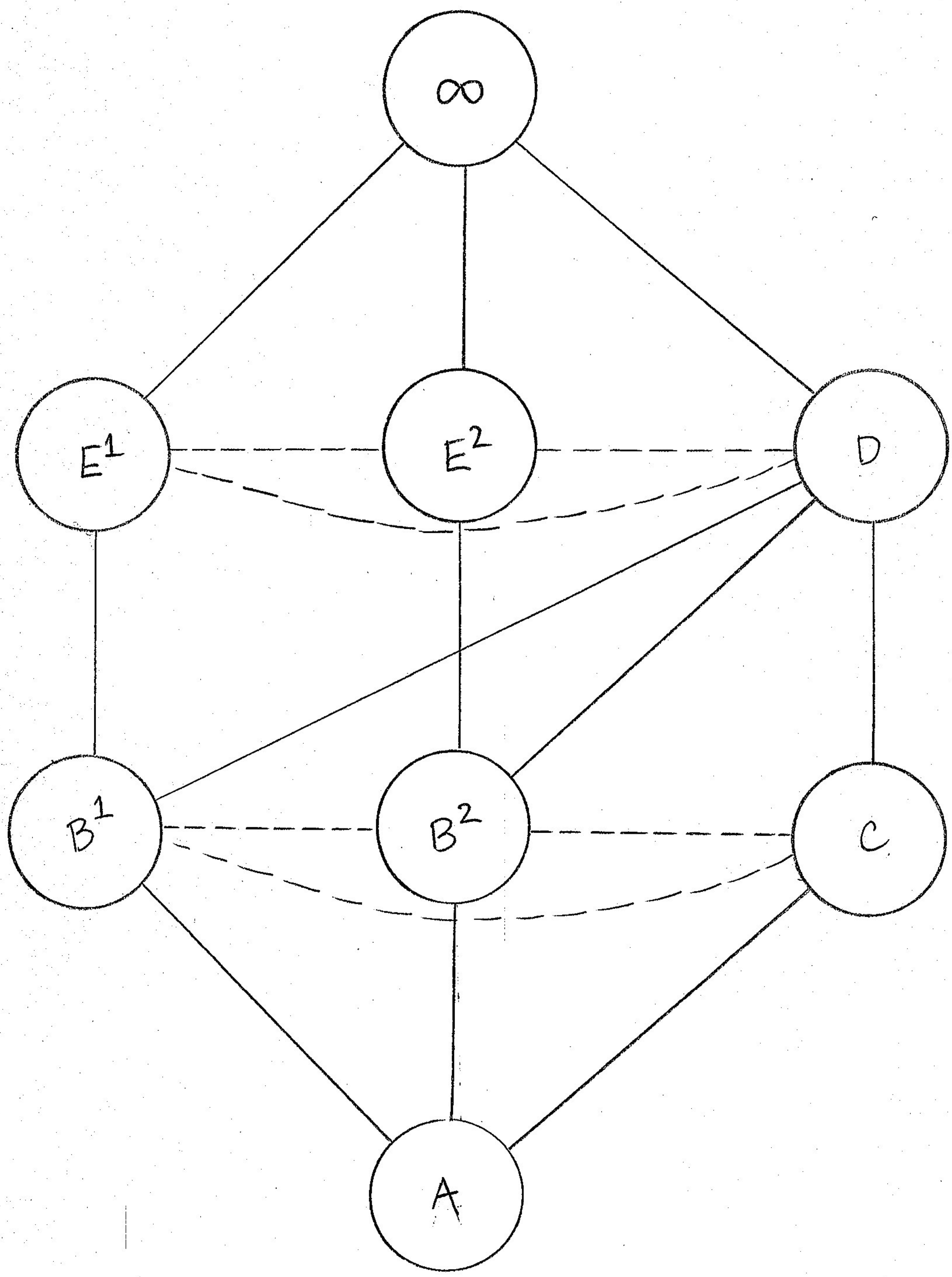


FIGURE 13