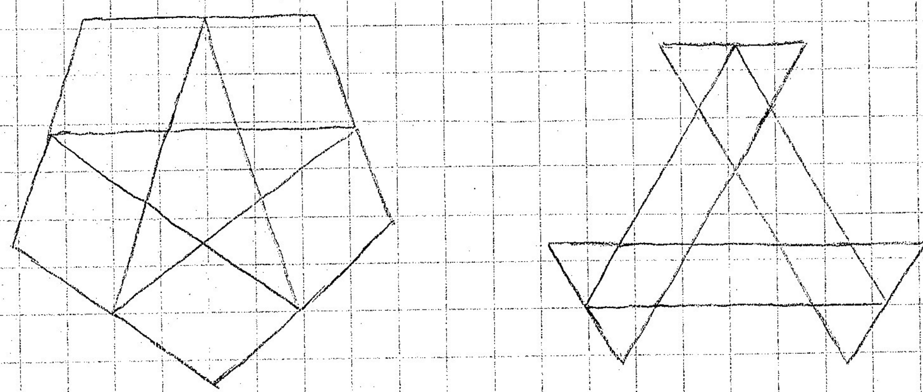
ICOSIDODECAHEDRAL STELLATIONS

Remarkably, fourteen uniform polyhedra are stellations of two cores consisting of an icosahedron and a dodecahedron. Many of these are among the most aesthetically pleasing of the uniform polyhedra, combining a richness of geometrical relationships with a simplicity of form.

We will consider the various polyhedra as members of $S[\underline{I}, \sigma \underline{D}]$ for some σ, so that the icosahedron is fixed. We will use metrical data for distances (such as that from the center of $\underline{D}$ to a vertex) as given, for example, in Williams' The Geometrical Foundation of Natural Structure, keeping in mind the fact that the edge lengths of an icosahedron and its dual dodecahedron are in the ratio $\tau : 1$. It is strongly recommended that you have a reference containing photographs or drawings of all the uniform polyhedra handy while reading what follows.

I. THE GREAT ICOSIDODECAHEDRON (94)

Looking closely at this polyhedron, one sees that it is "almost" a compound in the following sense. If the pentagrams are enlarged to pentagons as in Fig. 1(a), then in fact, the great dodecahedron (21) is obtained. Moreover, we see that the triangles may be "inscribed"



(a) (b)

Figure 1

in the stellation pattern for 28, a stellation of the icosahedron, as shown in Fig. 1(b). Thus, 94 is a compound of 21 and 28, except that twelve tiny inverted pentagonal pyramids must be inserted in the "dimples" of 28 to complete the facial plane of the pentagram.

We use these relationships to find σ such that 94 belongs to $S[\underline{I}, \sigma \underline{D}]$. We assume that the edge length of $\underline{D}$ is 1, so that the edge length of $\underline{I}$ is τ. (Of course, if the edge length of $\underline{D}$ is ε, all lengths in the following calculations may be scaled by a factor of ε.)

Now if 21 is a stellation of $\sigma \underline{D}$, it may be shown (see the discussion on dodecadodecahedral stellations) that the convex hull of 21 is an icosahedron of edge length $\tau^2 \sigma$, i.e., the icosahedron $\tau\sigma \underline{I}$. Analogously, if 28 is a stellation of $\underline{I}$, then its convex hull is a dodecahedron with edge length τ^2; i.e, the dodecahedron $\tau^2 \underline{D}$. This may easily be seen by examining the facial stellation pattern for the icosahedron (as given on p. 42 in PM).

It is evident from Fig. 1(a), (b) that the midpoints of the edges of $\tau\sigma \underline{I}$ and $\tau^2 \underline{D}$ are in fact coincident and comprise the vertices of 94. This implies, however, that $\tau\sigma \underline{I}$ and $\tau^2 \underline{D}$ are duals of each other. Now since $\underline{I}$ and $\underline{D}$ are dual, then $\tau\sigma \underline{I}$ and $\tau^2 \underline{D}$ can be dual only if $\tau\sigma = \tau^2$, and hence $\sigma = \tau$. Thus, we see that 94 belongs to $S[\underline{I}, \tau \underline{D}]$.

II THE SMALL DITRIGONAL ICOSIDODECAHEDRON (70)

It is evident that 70 is a truncated form of the first stellation of the icosahedron (26). In fact, it is a simple matter to

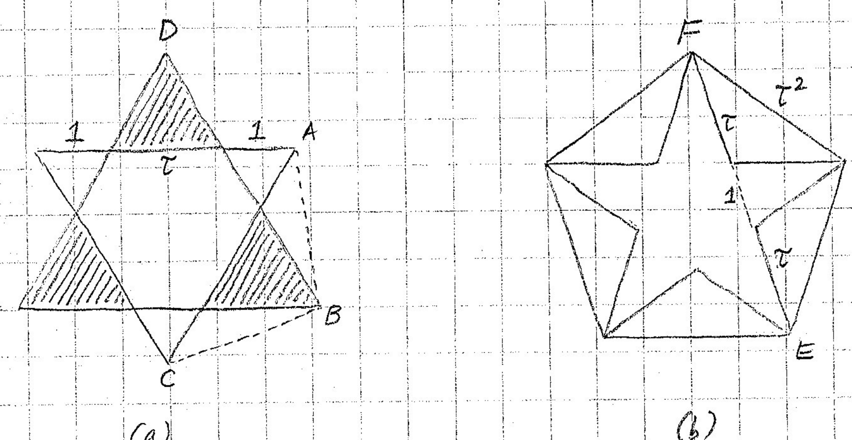


Figure 2

obtain the icosahedron of which 26 is a stellation from the triangular facial pattern (see p. 106 of PM) of 70; the "downward pointing" triangle in Fig 2(a) is the result. Moreover, triangle ABC is one of the sixty visible triangles of <u>26</u>.

It is also evident that the vertices of <u>70</u> are just the vertices of a dodecahedron, a pentagonal face of which circumscribes the pentagrams of <u>70</u> as in Fig. 2(b).

Now we seek σ so that <u>70</u> belongs to $S[\underline{I}, \sigma\underline{D}]$. For convenience, we assume that the edge length of the small <u>visible</u> equilateral triangles in <u>70</u> is τ (see Fig 2(a)). Because the larger triangles must be adjacent to the pentagrams, an edge such as $\overline{BD}$ in Fig 1(a) must be adjacent to an edge such as $\overline{EF}$ in Fig 2(b), and so the relative sizes of the various edges in Figs. 2(a),(b) are readily determined.

We see from Fig 2. that the edge length of $\underline{I}$ is $\tau+2$, while that of $\sigma\underline{D}$ is τ^2. Hence the edge length of $\sigma\underline{I}$ is $\sigma(\tau+2)$. Now $\sigma\underline{D}$ and $\sigma\underline{I}$ are dual, so we must have the edge lengths of these polyhedra in the ratio $1:\tau$, so that $\frac{\sigma(\tau+2)}{\tau^2}=\tau$. This results in the relationship $\sigma=\frac{\tau^3}{\tau+2}=\frac{1}{5}(3\tau+1)\approx 1.1708$.

One also sees, upon examining pp 52-53 of <u>Dual Models</u>, that <u>70</u> is a stellation of what Wenninger calls a "golden-section truncation" of the icosahedron.

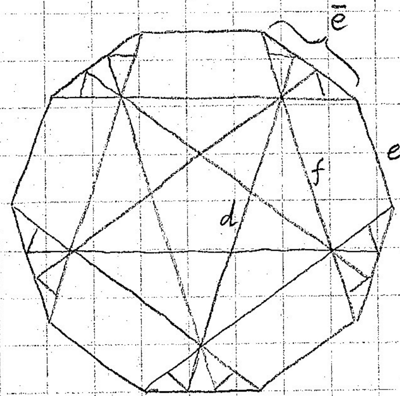
III THE GREAT DITRIGONAL ICOSIDODECAHEDRON (87)

Now 87 is very similar to 70. The vertices of 70 were, as we saw, the vertices of $\sigma\underline{D}$. The vertices of 87 are precisely the same as the vertices of 70, except that if 87 is a member of $S[\underline{I}, \sigma\underline{D}]$, we see that the vertices of 87 are the vertices of the final stellation of σD. This becomes evident by examining the pentagonal facial pattern for 87 on p. 135 of PM.

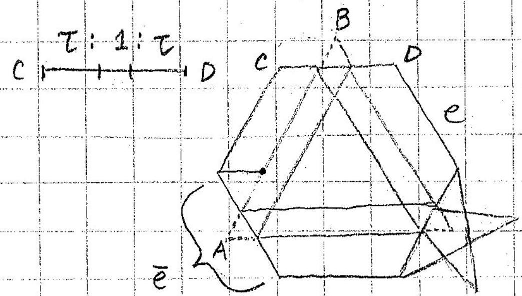
We base our calculations on the same triangle in Fig 1(a), so that the edge length of $\sigma\underline{I}$ is $\sigma(\tau+2)$. Now recall that the ratio between the edge length of $\sigma\underline{D}$ and the edge length of the dodecahedron enclosing the final stellation of $\sigma\underline{D}$ is $1:\tau^3$. But the edge length of the dodecahedron enclosing the final stellation of $\sigma\underline{D}$ is simply τ^2 (see Fig 2(b)), and hence the edge length of $\sigma\underline{D}$ is $\frac{1}{\tau}$. As before, since $\sigma\underline{D}$ and $\sigma\underline{I}$ are dual, their edge lengths must be in the ratio $1:\tau$, so that $\frac{\sigma(\tau+2)}{1/\tau} = \tau$, yielding $\sigma = \frac{1}{\tau+2} = \frac{1}{5}(3-\tau) \approx 0.2764$.

IV THE SMALL DODECICOSAHEDRON (90)

The geometrical relationships which allow us to determine σ such that 90 is in $S[\underline{I}, \sigma\underline{D}]$ are a bit more complicated than those just discussed. This is usually the case, however, when decagons which do not pass through the center of the polyhedron are involved.



(a)



(b)

Figure 3

We begin with a few diagrams, taken from p.141 of PM. In Fig 3(a), we let d be the edge length of the "base dodecahedron"; that is, the dodecahedron in whose facial planes lie the decagons of 90. f, e, and $\bar{e}$ are as indicated. Note that when the decagon is regular, we have $e = \bar{e}$; however, as we shall discuss a "perfect" version of 90 below, we will need to allow the possibility that $e \neq \bar{e}$.

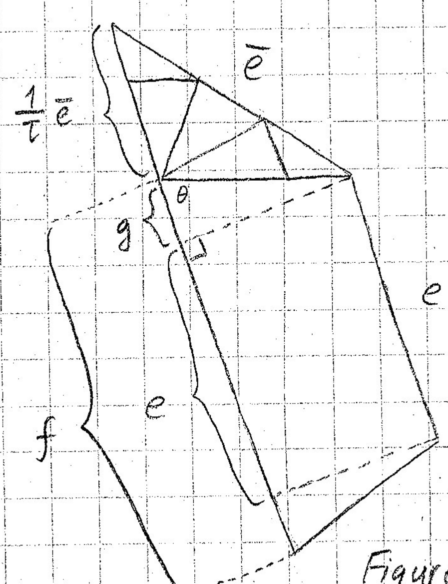


Figure 4

We consider in Figure 4 a magnification of a portion of Fig 3(a). Our immediate goal is to calculate d solely in terms of e and $\bar{e}$.

First, we find g. Note that $\theta = \frac{1}{5} = 72^\circ$, so that $\cos\theta = \frac{\tau - 1}{2}$. Using simple trigonometry, we see that

$$g = \frac{1}{\tau}\bar{e}\cos\theta = \frac{1}{\tau}\bar{e}\cdot\frac{\tau-1}{2} = \bar{e}\,\frac{2-\tau}{2}.$$

Therefore $f = e + 2g = e + 2\left(\bar{e} \cdot \frac{\tau - 2}{2}\right) = e + (2-\tau)\bar{e}$.

Now it is evident from Fig 3(a) that we have the relationship $f = \tau^2 d$, so that $d = \frac{1}{\tau^2} f = \frac{1}{\tau^2}\left(e + (2-\tau)\bar{e}\right) = (2-\tau)e + (5-3\tau)\bar{e}$.

Our next goal is to determine the edge length of the "base icosahedron"; that is, the icosahedron whose facial planes contain the hexagons of 90. In Fig. 3(b), we see that $\overline{AB}$ is an edge of this icosahedron (look at 90 for a moment to see why!). The lengths e and $\bar{e}$ in Fig 3(b) are determined by the adjacency of the decagons and hexagons in 90. Because of the "faceted" stars visible in 90, an edge of which $\overline{CD}$ is typical, the segments into which $\overline{CD}$ is divided must be in the ratio $\tau : 1 : \tau$ (as illustrated in Fig 3(b)).

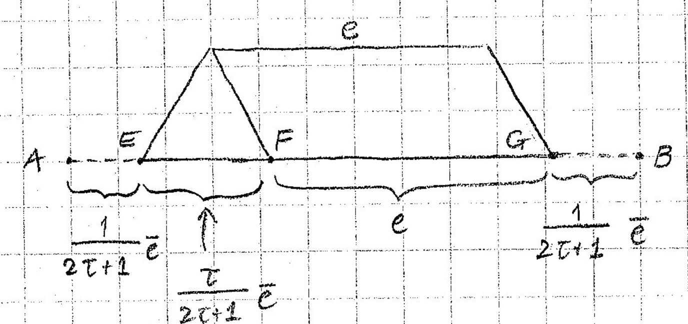


Figure 5

Now suppose that $\overline{AB}$ from Fig 3(b) is enlarged, as in Fig. 5. Both $\overline{AE}$ and $\overline{GB}$ are "middle" segments of $\overline{CD}$, and hence have length $\frac{1}{2\tau+1}\bar{e}$. $\overline{EF}$ is an "outer" segment of $\overline{CD}$, and hence has length $\frac{\tau}{2\tau+1}\bar{e}$. One sees from the parallelogram of which $\overline{FG}$ is a base that $\overline{FG}$ has length e. Thus, if $\overline{AB}$ has length i, we have $i = e + \left(\frac{1}{2\tau+1} + \frac{1}{2\tau+1} + \frac{\tau}{2\tau+1}\right)\bar{e}$

$= e + \frac{\tau+2}{2\tau+1}\bar{e} = e + (3\tau - 4)\bar{e}$

We are now in a position to determine σ such that $\underline{90}$ belongs to $\mathcal{S}[\underline{I}, \sigma\underline{D}]$. We know that the edge length of $\sigma\underline{D}$ is simply d. The edge length of $\underline{I}$ is i, so that the edge length of $\sigma\underline{I}$ is σi. As in the previous two cases, we see that $\sigma\underline{D}$ and $\sigma\underline{I}$ are dual, and hence $\frac{\sigma i}{d} = \tau$, which implies $\sigma = \frac{\tau d}{i}$.

A. THE REGULAR CASE

As we remarked earlier, in $\underline{90}$ the decagons are regular, so that $e = \bar{e}$. Substituting e for $\bar{e}$ in our relationships for d and i yields $d = (7-4\tau)e$ and $i = (3\tau-3)e$. Thus, we have $\sigma = \frac{\tau d}{i} = \frac{\tau(7-4\tau)e}{(3\tau-3)e} = \frac{\tau(7-4\tau)}{3\tau-3} = \frac{3-\tau}{3} \approx 0.4607$. As a result, we see that $\underline{90}$ belongs to $\mathcal{S}[\underline{I}, \frac{3-\tau}{3}\underline{D}]$.

B. THE PERFECT CASE.

Just as several uniform polyhedra with octahedral symmetry have "perfect" versions whose algebraic representations are simpler than those of their uniform counterparts, such is the case with many of the uniform polyhedra with icosahedral symmetry. As we shall see, $\underline{90}$ is a case in point.

How shall we characterize perfect versions of uniform polyhedra with icosahedral symmetry? We shall allow decagons and hexagons to have edge lengths which alternate in the ratio, as you might guess, of $\tau : 1$. We shall see abundant evidence why this should be the required characterization.

Indeed, "perfect" decagons are very easy to construct. The reader may easily verify that a perfect decagon may be produced simply by dividing the edges of a pentagon into equal thirds (Fig. 6).

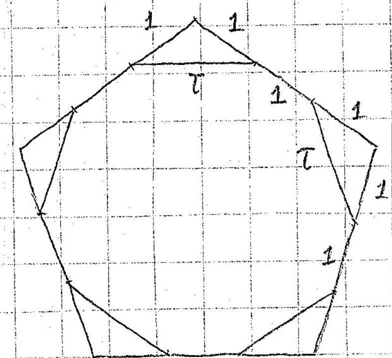


Figure 6

With this in mind, we may insure that our decagons are perfect be requiring that $e = \tau\bar{e}$ or $\bar{e} = \tau e$. As it happens, the former gives the most elegant result. Indeed, substituting $\tau\bar{e}$ for e in our relationships for d and i yields $d = (2-\tau)\tau\bar{e} + (5-3\tau)\bar{e} = (4-2\tau)\bar{e}$ and $i = \tau\bar{e} + (3\tau - 4)\bar{e} = (4\tau - 4)\bar{e}$. Thus, we have $\sigma = \frac{\tau d}{i} = \frac{\tau(4-2\tau)\bar{e}}{(4\tau-4)\bar{e}} = \frac{\tau(4-2\tau)}{4\tau-4} = \frac{2\tau-2}{4\tau-4} = \frac{1}{2}$. Perfect! Hence, we see that the perfect version of $\underline{90}$ belongs to $S[\underline{I}, \frac{1}{2}\underline{D}]$.

One can readily create a physical model of $\underline{90}$ by modifying the diagrams in Fig 3 so that $e = \tau\bar{e}$, thereby creating the appropriate facial patterns. The procedure, of course, is identical to that given in PM for $\underline{90}$.

V THE TRUNCATED GREAT ICOSAHEDRON (95)

Unfortunately, there appears to be no simple, elegant way to determine σ such that 95 is in $\mathcal{S}[\underline{I}, \sigma \underline{D}]$. There is, however, a rather straightforward algebraic approach. We will call this approach the "routine method", as it can faithfully be used (even in the cases already discussed) to produced the desired σ for most uniform polyhedra.

So suppose, for convenience, that the edges of 95 have length 1. Then the radius of the circumsphere is $\rho = \frac{1}{2}\sqrt{\frac{29-9\sqrt{5}}{2}}$ (see PM, p. 148, for example, where this radius is given for an edge of length 2). We set two intermediate goals: find the distance r_6 from the center of 95 to the center of a hexagonal face, and the distance r_5 from the center of 95 to the center of a pentagrammatic face.

We begin with a few diagrams. We see in Fig 7(a) a

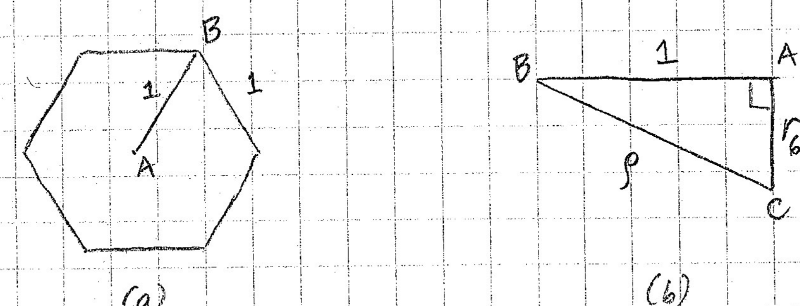


Figure 7

hexagonal face of 95. In Fig 7(b), C is the center of 95, so that with A and B from Fig 7(a), a right triangle BAC is formed, with r_6 and ρ as labeled. We see from the Pythagorean theorem that $r_6^2 + 1 = \rho^2$, so that $r_6^2 = \rho^2 - 1 = \frac{15-9\tau}{4}$ (the details of such calculations are left to the reader).

We analogously calculate r_5. In Fig 8(a), we see a pentagrammatic

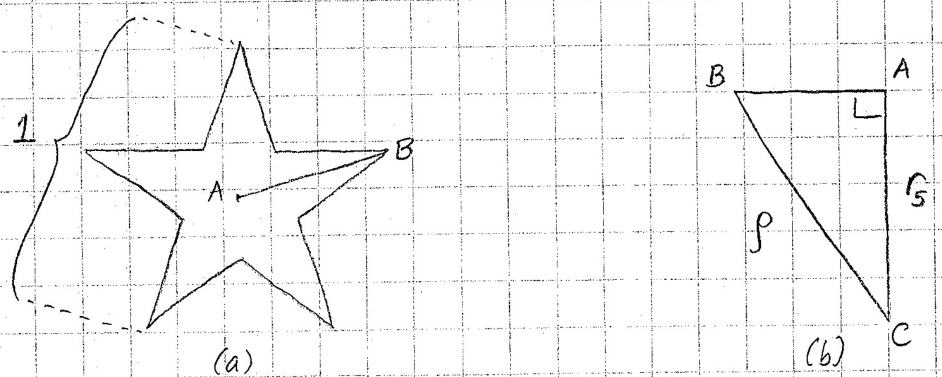


Figure 8

face of 95. By using some geometry of regular pentagons and pentagrams, we see that if one edge of the star has length 1, then we have $\overline{AB}^2 = \frac{3-\tau}{5}$. Fig 8(b) is analogous to Fig 7(b), where C is the center of 95. Thus, $r_5^2 = \rho^2 - \overline{AB}^2 = \frac{83-41\tau}{20}$. So if we define $\hat{\delta} := \frac{r_5}{r_6}$, we find that

$$\hat{\delta}^2 = \left(\frac{r_5}{r_6}\right)^2 = \frac{83-41\tau}{5(15-9\tau)}.$$

Now that we have achieved our intermediate goals, we seek an analogous ratio for I and D to which we might compare $\hat{\delta}$. Using data such as that given in Williams, we find that the ratio δ of the distance from the center of $\underline{D}$ to the center of a face of $\underline{D}$, to the distance from the center of $\underline{I}$ to the center of a face of $\underline{I}$, satisfies the relationship $\delta^2 = \frac{3}{\sqrt{5}\tau}$.

Having found δ, we see that σ is given by the ratio $\frac{\hat{\delta}}{\delta}$. Our method here differs from the preceding in that we used distances from the centers of $\underline{I}$ and $\underline{D}$ to the centers of their faces rather than edge lengths. Given our method of calculating δ and $\hat{\delta}$, we think of these ratios as being relative to a "fixed" icosahedron, so that our definition of σ makes geometrical sense. Carrying out the tedious computational details yields

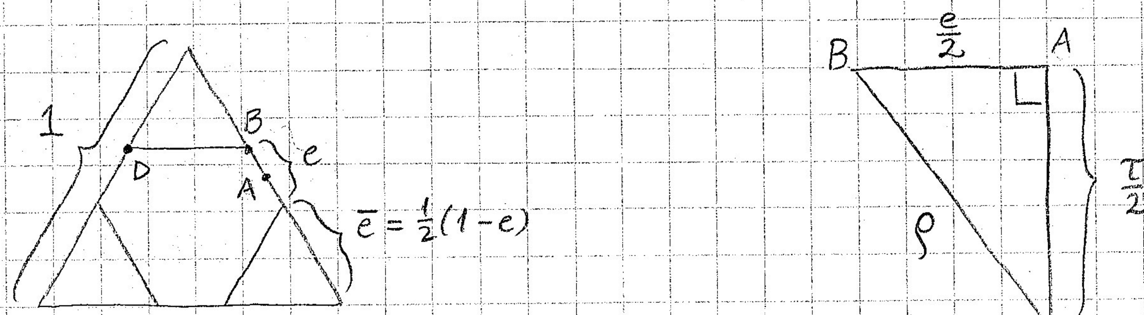
$$\sigma = \frac{\hat{\delta}}{\delta} = \sqrt{\frac{83-41\tau}{5(15-9\tau)}} \cdot \sqrt{\frac{\sqrt{5}\tau}{3}} = \frac{1}{3}\sqrt{26+25\tau} \approx 2.7172 \approx 1.0379\tau^2.$$

Happily, $\underline{95}$ admits an elegant "perfect" version. If one "completes" the truncation of the great icosahedron so that the pentagrams are positioned so that their vertices touch (as in $\underline{80}$), we see that the hexagons thereby formed have edges which alternate in the ratio $\tau : 1$. In this case, this perfect polyhedron belongs to $S[\underline{I}, \tau^2 \underline{D}]$. It is not too closely related to $\underline{80}$, however, as $\underline{80}$ is a dodecadodecahedral stellation, while $\underline{95}$ is not.

VI. THE TRUNCATED ICOSAHEDRON (9)

We may apply a similar analysis to the truncated icosahedron and a few of its "relatives". Assume that the icosahedron being truncated is simply I, whose edge length is 1.

In Fig. 9(a), we see a triangular face of I. We allow the

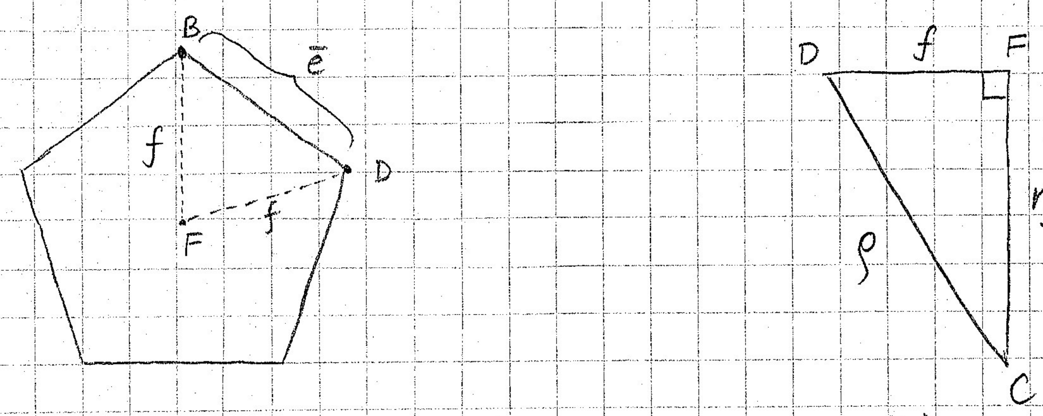


(a) Figure 9 (b)

division of the edge to be other than equal thirds, although this particular case will certainly be discussed. Note that A is the center of an icosahedral edge.

In Fig 9(b), C is the center of the icosahedron, while A and B are as in Fig. 9(a). As before, one may consult a source such as Williams (p.67) to see that $\overline{AC} = \frac{\tau}{2}$. It easily follows from the Pythagorean theorem that $\rho^2 = \frac{1}{4}(e^2 + \tau^2)$.

In Fig 10(a), we see a pentagonal face of the truncated icosahedron, whose edge BD is the same as that of the hexagon in Fig. 9(a). Note that $\overline{BD} = \bar{e}$. The center of this face is



(a) Figure 10 (b)

denoted by F. One may verify from the geometry of the pentagon that the length $f = \overline{FD}$ satisfies $f^2 = \frac{1}{5}\bar{e}^2(\tau+2)$.

In Fig 10(b), C is again the center of $\underline{I}$, so that r_5 is the distance from C to the center of the pentagonal faces. One readily sees that $r_5^2 = \rho^2 - f^2$, which, using our previous relationships for ρ and f, yields $r_5^2 = \frac{1}{4}(e^2+\tau^2) - \frac{1}{5}\bar{e}^2(\tau+2)$.

Of course we must compare r_5 to the distance from C to the center of the faces of the dodecahedron $\underline{D}$ which is dual to $\underline{I}$. If we call this distance r, we may again retrieve data which gives the relationship $r^2 = \frac{4\tau+3}{20}$. Thus, by defining $\sigma := \frac{r_5}{r}$, we see that the truncated icosahedron belongs to $S[\underline{I}, \sigma\underline{D}]$.

Case 1: $e = 0$, $\bar{e} = \frac{1}{2}$.

In this case, we "fully" truncate $\underline{I}$ to obtain an icosidodecahedron. A straight-forward calculation reveals that $\sigma = 1$ in this case, which makes sound geometrical sense since we know that the volume common to an icosahedron and its dual dodecahedron is enclosed by an icosidodecahedron. Thus 12, the icosidodecahedron, belongs to $S[\underline{I}, \underline{D}]$.

Case 2: $e = \bar{e} = \frac{1}{3}$

In this case, the "usual" truncated icosahedron, 9, results. One verifies in this case that $r_5^2 = \frac{41\tau + 42}{180}$, so that

$$\sigma = \sqrt{\frac{41\tau+42}{180} \cdot \frac{20}{4\tau+3}} = \sqrt{\frac{26-9\tau}{9}} = \frac{5-\tau}{3}.$$

Case 3: $e = \frac{\tau}{\tau+2}$, $\bar{e} = \frac{1}{\tau+2}$.

In this case, the "golden-section" truncation of the icosahedron results (see Dual Models by Wenninger, pp. 52-3). Although the calculations are somewhat tedious, some diligence yields the result that $\sigma = \frac{r_5}{r} = \frac{1}{5}(3\tau+1)$. This result confirms the observation made in (II) above that 70 is in fact a stellation of the golden-section truncation of the icosahedron.

VII. THE REMAINING POLYHEDRA

There are three other uniform polyhedra which may be analyzed with the methods described thus far. The first is the truncated dodecahedron (10). One finds that 10 belongs to $S[\underline{I}, \frac{7\tau-6}{5}\underline{D}]$; the details of this calculation shall be omitted. Unfortunately, requiring that the truncation occur so that perfect decagons result does not appear to yield any algebraic simplification.

Again, using similar methods, we see that the great dodecicosahedron (<u>101</u>) belongs to $S[\underline{I}, (\tau+\frac{1}{3})\underline{D}]$. Likewise, we see that the quasitruncated great stellated dodecahedron (<u>104</u>) is a member of $S[\underline{I}, \frac{\tau+7}{11}\underline{D}]$. Quite an interesting juxtaposition of "7" and "11"!

We have at this point described all the icosidodecahedral uniform polyhedra but four. These four have "equatorial" faces; that is, faces which pass through the center of the polyhedron. As a result, they must belong to either $S[\underline{I}, 0\underline{D}]$, or $S[\underline{I}, \infty\underline{D}] = S[0\underline{I}, \underline{D}]$.

We first discuss the two which belong to $S[\underline{I}, 0\underline{D}]$; namely, the small icosihemidodecahedron (<u>89</u>) and the great icosihemidodecahedron (<u>106</u>). As it happens, <u>106</u> is related to <u>89</u> in the same way that <u>107</u> is related to <u>91</u> (see the discussion of dodecadodecahedral polyhedra). That is, if the equatorial decagons of <u>89</u> are enlarged to decagrams, and if the triangular faces are suitably enlarged, then <u>106</u> is the result. Thus, <u>106</u> may be considered, in an approximate sense, as a "stellation" of <u>89</u>.

The remaining two polyhedra, the small dodecahemicosahedron (100) and the great dodecahemicosahedron (102) belong to $S[O\underline{I}, \underline{D}]$. Again, we may consider 102 as an enlargement of 100 in that if the hexagons of 100 are suitably enlarged, and if the pentagrams are extended to much larger pentagons, then the result is 102. This is

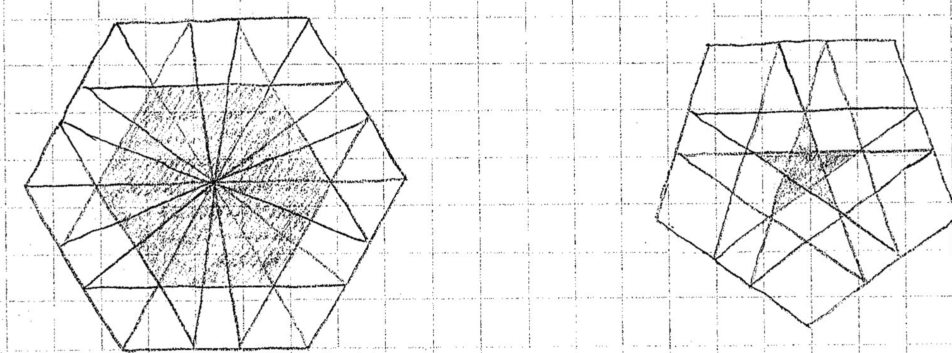


Figure 11

easily seen in Fig. 11, where the facial diagrams for 102 are illustrated (taken from PM, p. 158), and the corresponding "embedded" faces of 100 are shaded.