Non-convex non-snub uniform polyhedra and their duals.

I. Notation.

Let $\mathbb{P}$ represent the positive reals including zero, and $\mathbb{P}^x$ those excluding zero. Let $\overline{\mathbb{P}}$ represent $\mathbb{P} \cup \{\infty\}$.

Let $\underline{I}$ represent a regular tetrahedron, $\underline{O}$ represent the octahedron whose vertices are midpoints of the edges of $\underline{I}$, $\underline{C}$ represent the cube dual to $\underline{O}$ in the sense that its edges are perpendicular bisectors of the edges of $\underline{O}$, $\underline{I}$ represent an icosahedron five of whose faces are included in the facial planes of $\underline{I}$ (note that there are two possibilities), and $\underline{D}$ represent the dual of $\underline{I}$ in the sense just described.

If $\underline{P}$ is a polyhedron and $\rho \in \mathbb{P}^x$ is given, then $\rho\underline{P}$ is that polyhedron obtained by scaling $\underline{P}$ by a factor of ρ. $\rho\underline{P}$ is scaled in such a way that any ray emanating from the center of $\underline{P}$ which passes through a vertex of $\underline{P}$ also passes through a vertex of $\rho\underline{P}$.

Let $\underline{P}$ and $\underline{Q}$ be two polyhedra, and let $\mathcal{G}$ be the group of symmetries common to both. (We will usually consider cases where $\underline{P}$ and $\underline{Q}$ possess the same symmetries.) The facial planes of $\underline{P}$ and $\underline{Q}$ decompose space into a finite number of closed cells (two cells may share a boundary which is a closed subset of a plane), some of which are bounded, some of which are not. A subset of such bounded cells whose symmetry group is $\mathcal{G}$

is called a stellation of the compound of $\underline{P}$ and $\underline{Q}$. The set of all such stellations is denoted by $S[\underline{P}, \underline{Q}]$.

A polyhedron whose vertices are precisely the vertices of $\underline{P}$ and $\underline{Q}$ together is called a faceting of the compound of $\underline{P}$ and $\underline{Q}$. The set of all such facetings is denoted by $\mathcal{F}[\underline{P}, \underline{Q}]$. Of course this and the previous definition may be extended to more than two polyhedra or restricted to a single polyhedron.

II. Generalized Stellations

Members of $S[\underline{C}, \underline{O}]$ are stellations of the cuboctahedron, members of $S[\underline{C}, \frac{3}{4}\underline{O}]$ are stellations of the truncated octahedron, and members of $S[\underline{C}, \frac{\sqrt{2}+1}{2}\underline{O}]$ are stellations of the truncated cube. The octahedron $\frac{1}{2}\underline{O}$ is such that its vertices are the barycenters of the faces of $\underline{C}$; when $0 < \rho < \frac{1}{2}$, $\rho\underline{O}$ lies entirely within $\underline{C}$. Therefore, when $0 < \rho \leq \frac{1}{2}$, we call members of $S[\underline{C}, \rho\underline{O}]$ generalized stellations – for the facial planes of $\underline{C}$ and $\rho\underline{O}$ cannot be described as the facial planes of a single convex polyhedron. Likewise, the vertices of $\underline{C}$ are precisely the barycenters of the faces of $\frac{3}{2}\underline{O}$, so we call members of $S[\underline{C}, \rho\underline{O}]$ generalized stellations when $\rho \geq \frac{3}{2}$. When $\rho = \frac{\sqrt{2}-1}{2}$, we find that the largest (with respect to set inclusion) stellation in $S[\underline{C}, \rho\underline{O}]$ is in fact the

uniform polyhedron 92 (where underlined numbers refer to model numbers in Wenninger's Polyhedron Models). It is evident, in fact, that all the uniform polyhedra are generalized stellations of a compound of a finite number of convex polyhedra.

III. The cases $\rho = 0$ and $\rho = \infty$.

When ρ is very small, the opposite parallel facial planes of $\rho\underline{O}$ in the compound of $\underline{C}$ and $\rho\underline{O}$ are so close together as to appear coincident. In this case, all that "appears" are the facial planes of $\underline{C}$ and the planes including the four equatorial hexagons of $\underline{O}$. This is the interpretation which we assign to $S[\underline{C}, 0\underline{O}]$. Similarly, for any two polyhedra $\underline{P}$ and $\underline{Q}$, we interpret $S[\underline{P}, 0\underline{Q}]$ to be the "limiting case" of $S[\underline{P}, \rho\underline{Q}]$ as ρ decreases towards zero. We see that the uniform polyhedron 78 is in $S[\underline{C}, 0\underline{O}]$.

When ρ is large, we observe that members of $S[\underline{C}, \rho\underline{O}]$ are simply members of $S[\frac{1}{\rho}\underline{C}, \underline{O}]$ scaled by a factor of ρ. Using poetic license and taking "the limit" as ρ increases without bound, we define $S[\underline{C}, \infty\underline{O}]$ to be the generalized stellations $S[0\underline{C}, \underline{O}]$.

We may similarly discuss $\mathcal{F}[\underline{C}, 0\underline{O}]$ and $\mathcal{F}[\underline{C}, \infty\underline{O}]$, so that, for example, $\mathcal{F}[\underline{C}, 0\underline{O}]$ are those polyhedra whose vertices are those of $\underline{C}$

along with the barycenter of $\underline{C}$. Of course, for any polyhedra $\underline{P}$ and $\underline{Q}$, we may similarly define $S[\underline{P}, \infty\underline{Q}]$, $\mathcal{F}[\underline{P}, 0\underline{Q}]$, and $\mathcal{F}[\underline{P}, \infty\underline{Q}]$.

IV. Stellations of the dodecahedron

The facial planes of $\underline{D}$ divide space into eight different types of regions - four bounded and four unbounded. We put $\underline{D}_0 := \underline{D}$. The cells visible in the small stellated dodecahedron ($\underline{20}$) are denoted by $\underline{D}_1$; those visible in the great dodecahedron ($\underline{21}$) are denoted by $\underline{D}_2$; and those visible in the great stellated dodecahedron ($\underline{22}$) are denoted by $\underline{D}_3$. The unbounded cells on top of $\underline{D}_3$ are denoted by $\underline{D}_4$, and the unbounded cells on top of the $\underline{D}_4$ are denoted by $\underline{D}_5$. There are two types of unbounded cells on top of $\underline{D}_5$; the unbounded trihedral [pentahedral] cups whose vertices lie on the 3-fold [5-fold] axes of symmetry are denoted by $\underline{D}_6$ [$\underline{D}_7$]. Thus, if $\underline{D}_i \prec \underline{D}_j$ means "the cells $\underline{D}_j$ are on top of the cells $\underline{D}_i$", we have

$$\underline{D}_0 \prec \underline{D}_1 \prec \underline{D}_2 \prec \underline{D}_3 \prec \underline{D}_4 \prec \underline{D}_5 \prec \underline{D}_6, \quad \underline{D}_5 \prec \underline{D}_7.$$

For $\rho \in \mathbb{P}$, we denote the analogous cells in $S[\rho\underline{D}]$ by $\rho\underline{D}_i$, $0 \le i \le 7$.

We may consider $S[\underline{D}, \rho\underline{D}]$ when $\rho \in \mathbb{P}$. In this case, the cells $\underline{D}_i \cap \rho\underline{D}_j$, $0 \le i, j \le 7$, completely decompose space (although some intersections may be empty or degenerate into planar regions). For brevity,

we use the abbreviation "ij" to represent the cells $\underline{D}_i \cap \rho \underline{D}_j$ when discussing such generalized stellations.

V THE DIDODECAHEDRAL UNIFORM POLYHEDRA

There are exactly six uniform polyhedra which may be considered as generalized stellations of the form $\mathcal{S}[\underline{D}, \rho \underline{D}]$ with $\rho \in \mathbb{P}$.

A. The Dodecadodecahedron (73).

This polyhedron may be viewed as a truncation of the great dodecahedron. It is a member of $\mathcal{S}[\underline{D}, \tau \underline{D}]$, where τ is the golden ratio; i.e, $\tau = \frac{1+\sqrt{5}}{2}$. Recalling the abbreviation $ij := \underline{D}_i \cap \tau \underline{D}_j$, we find that 73 is composed of the cells 00, 10, and 20.

B. The Truncated great dodecahedron (75).

This polyhedron belongs to $\mathcal{S}[\underline{D}, \rho \underline{D}]$, where $\rho = \frac{2}{\sqrt{5}}\left(1+2\sin\frac{\pi}{5}\right)$. As in the previous case, 75 is composed of the cells 00, 10, and 20.

C. The Ditrigonal dodecahedron (80).

This polyhedron is a member of $\mathcal{S}[\underline{D}, \tau^3 \underline{D}]$ It is composed of the cells 00, 10, 20, 30, 50, and 70. Thus, we see that the removal of those parts of $\tau^3 \underline{D}_0$ which lie in the unbounded cells D_4 are responsible for the deep hexahedral cups in this polyhedron.

D. The Quasitruncated small stellated dodecahedron (97).

This polyhedron belongs to $S[\underline{D}, \sqrt{5}\tau^3\underline{D}]$ It is comprised of the dodecahedron $\sqrt{5}\tau^3 D_0$ together with the cells 41 and 51. Thus, if we remove from the small stellated version of $\sqrt{5}\tau^3\underline{D}$ those parts of $\sqrt{5}\tau^3 D_1$ which belong to $\underline{D}_7$, we obtain the polyhedron 97.

E. The Small dodecahemidodecahedron (91).

This polyhedron is a member of $S[\underline{D}, 0\underline{D}]$. Recall that $0\underline{D}_6$ and $0\underline{D}_7$ are the only non-degenerate cells in $S[0\underline{D}]$. 91 is composed of the single cell 07.

F The Great dodecahemidodecahedron (107).

This polyhedron belongs to $S[\underline{D}, 0\underline{D}]$. It is, in a sense, an "enlargement" of 91 – if we enlarge the equatorial decagons of 91 to decagrams and enlarge the pentagons to pentagrams, we obtain 107. As a generalized stellation, it is composed of the cells 06, 07, 16, and 26.

Alternatively, we may view this polyhedron as D_0 together with those parts of the great dodecahedron (i.e, D_0, D_1, and D_2) belonging to $0\underline{D}_6$. Thus, adding 17 and 27 to 107 "fills" the pentahedral cups and yields a great dodecahedron.

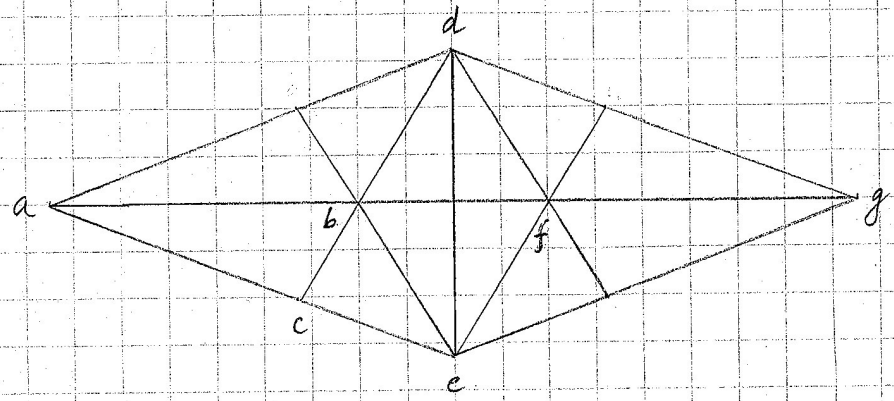
VI. Duality

We base our discussion of duality on the standard procedure of obtaining duals by polar reciprocation. We begin by discussing the dual of $\underline{92}$, which belongs to $S[\underline{C}, \rho\underline{O}]$, where $\rho = \frac{\sqrt{2}-1}{2}$. To obtain a vertex set for this dual, we find the poles of the facial planes of $\underline{C}$ and $\rho\underline{O}$. We choose as our radius of inversion $\sqrt{\rho}$, so that the facial planes of $\underline{C}$ invert to the vertices of $\rho\underline{O}$, and the facial planes of $\rho\underline{O}$ invert to the vertices of $\underline{C}$. Thus, we see that the dual of $\underline{92}$ belongs to $\mathcal{F}[\underline{C}, \rho\underline{O}]$. This is clear upon examining the dual in Photo 34 (all such references indicate photographs in Wenninger's Dual Models), keeping in mind that $\rho \approx 0.207$.

VII. An Example

We now wish to consider the dual of $\underline{73}$, the medial rhombic triacontahedron (Photo 27). Recall that $\underline{73}$ belongs to $S[\underline{D}, \tau\underline{D}]$. By choosing our radius of inversion to be $\sqrt{\tau}$, and recalling that $\underline{I}$ is dual to $\underline{D}$, we see that the m.r.t. is a member of $\mathcal{F}[\underline{I}, \tau\underline{I}]$. This is evident upon examining Photo 27, where the vertices of $\underline{I}$ are the "hidden vertices" of the m.r.t. lying directly beneath the vertices of $\tau\underline{I}$.

We may use this simple relationship between the two icosahedral sets of vertices for the m.r.t. to construct a diagram for the rhombic faces of the m.r.t. This is done in the figure below.



The following simple relationships are obtained:

$$\frac{ag}{bf} = \tau^3, \quad \frac{de}{bf} = \frac{ae}{ac} = \tau.$$

These ratios are sufficient to determine the figure. Several comments are in order:

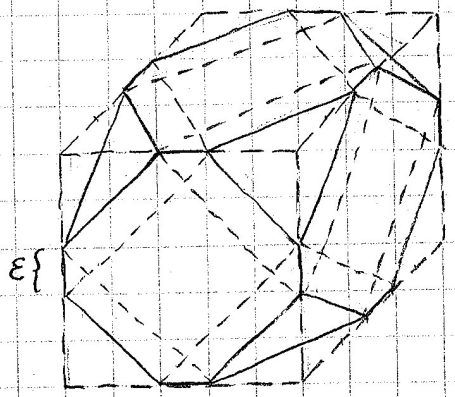
1. ag is obtained as a diagonal of a pentagon formed by taking the five vertices adjacent to a given vertex of $\tau\underline{\underline{I}}$;
2. de is an edge of $\underline{\underline{I}}$;
3. c, b, and d are collinear (all lying on a segment which is the intersection of two rhombic faces);
4. bdfc is a face of the rhombic triacontahedron of which the m.r.t. is a stellation

VIII. HEMIPOLYHEDRA

With these ideas in mind, we describe a procedure for finding duals for the hemipolyhedra, a procedure which differs from Wenninger's (see Dual Models, p. 101 ff). We illustrate the process by finding a dual for 78.

Recall that 78 belonged to $S[\underline{C}, 0\underline{O}]$, which we could "approximate" by $S[\underline{C}, \rho\underline{O}]$ for ρ near zero. Thus, our attack is to approximate 78 by some $\underline{P}_\rho$ in $S[\underline{C}, \rho\underline{O}]$, find the dual of $\underline{P}_\rho$ in $\mathcal{F}[\underline{C}, \rho\underline{O}]$, and examine these duals as ρ decreases to zero.

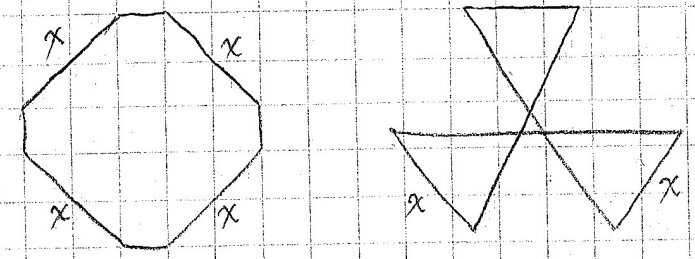
We approximate 78 by the following polyhedron $\underline{P}_\rho$ in $S[\underline{C}, \rho\underline{O}]$:



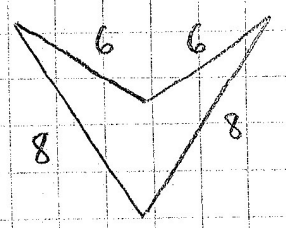
In this polyhedron, the facial planes of $\rho\underline{O}$ intersect the faces of $\underline{C}$ in eight lines: four form four edges of the facial octagons, the other four are interior to this octagon and are indicated by dashed lines in the above figure. Note that as ρ tends towards zero, so does ε (as indicated above); that is, the octagons eventually "degenerate" into the squares of 78.

In order that $\underline{P}_p$ bear some resemblance to $\underline{78}$, we remove the "corners" from $\underline{C}$; more precisely, we remove those parts of $\underline{C}$ which belong to the infinite trihedral cups opposite the vertices of the stella octangula which is a stellation of $p\underline{Q}$. This results in the deep trihedral pockets resembling those in $\underline{78}$

Upon further scrutiny, we see that the faces of $\underline{P}_p$ are the six octagons and eight "crossed" hexagons (where edges labelled "x" all have the same length) as illustrated below:

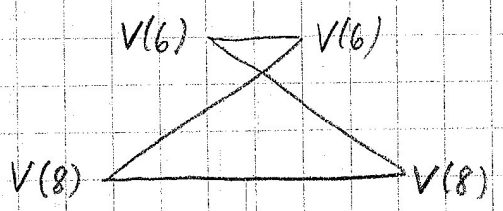


The small triangle in the center of the hexagonal face is in fact a face of the stella octangula which is in $S[p\underline{Q}]$. A verlex figure for $\underline{P}_p$ is roughly given as follows:



As a result, a face of the dual of $\underline{P}_p$ is approximately given by

the following (where the notation is consistent with Wenninger's notation for the faces of duals):



The two vertices labelled "V(8)" are adjacent in $\underline{C}$, and the two labelled "V(6)" are adjacent in $\rho\underline{Q}$. Quantitative details are not critically important, since when ρ decreases to zero, the "V(6)" vertices all tend towards the barycenter of the cube. Thus, as ρ decreases to zero, these duals "converge" to a polyhedron consisting of twelve isosceles triangles, each of whose vertices consist of two adjacent vertices of $\underline{C}$ along with the barycenter of $\underline{C}$. This polyhedron belongs to $\mathcal{F}[\underline{C}, 0\underline{Q}]$, and it is this polyhedron which we consider to be the dual of $\underline{78}$.