DODECADODECAHEDRAL STELLATIONS

Of course, there is no particular reason why the two core polyhedra need to be different, as was the case with the cube and octahedron of the previous section. Here, we discuss stellations which arise from two regular dodecahedral cores which are scaled with respect to one another.

I. SPATIAL DECOMPOSITION BY $\underline{D}$.

Let $\underline{D}$ represent a regular dodecahedron. As with the cube and octahedron, the facial planes of $\underline{D}$ divide space into several regions; in this case, four types of bounded regions and four types of unbounded regions, all of which shall now be described.

$\underline{D}_0$ is just the dodecahedral core. The twelve pentagonal pyramids visible on the small stellated dodecahedron (20 in PM (Polyhedron Models, by Wenninger)) are denoted by $\underline{D}_1$, the thirty tetrahedra visible on the great dodecahedron (21 in PM) are denoted by $\underline{D}_2$, and the twenty triangular bipyramids visible on the great stellated dodecahedron (22 in PM) are denoted by $\underline{D}_3$. The thirty unbounded cells

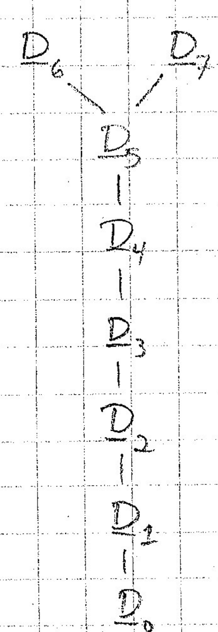


Figure 1

with rhombic cross-sections on top of the $\underline{D}_3$ are denoted by $\underline{D}_4$, and the sixty unbounded cells on top of the $\underline{D}_4$ are denoted by $\underline{D}_5$. There are two types of unbounded cells on top of the $\underline{D}_5$; the twenty trihedral "cups" whose vertices lie on the threefold axes of symmetry of $\underline{D}$ are denoted by $\underline{D}_6$, while the twelve pentahedral cups whose vertices lie on the fivefold axes of symmetry of $\underline{D}$ are denoted by $\underline{D}_7$. A cell adjacency diagram for $\underline{D}$ is given in Figure 1.

II. Transitional Values of σ

As with the cuboctahedral case, we now seek values of σ for which there are degenerate cells among the "ij"; that is, the various cells formed by the intersections of cells bounded by facial planes of $\underline{D}$ and cells bounded by the facial planes of $\sigma\underline{D}$.

As usual, $\sigma = 0$ and $\sigma = \infty$ are transitional. The smallest nonzero transitional value is $\sigma = \frac{1}{\tau^3}$, where τ is the golden ratio (i.e, $\tau = \frac{1+\sqrt{5}}{2}$). In this case, we find that the final stellation (see $\underline{22}$ in PM) of $\frac{1}{\tau^3}\underline{D}$ is exactly inscribed in $\underline{D}$; that is, the apices of the bipyramids $\frac{1}{\tau^3}\underline{D}_3$ are precisely the vertices of $\underline{D}$. By scaling <u>both</u> of these dodecahedra by a value of τ^3, we likewise find that the final stellation of $\underline{D}$ is exactly inscribed in $\tau^3\underline{D}$, and hence $\sigma = \tau^3$ is a transitional value as well. This "trick" may be performed for <u>any</u> transitional value of σ, so that if σ is such a value, then so is $\frac{1}{\sigma}$. This, happily, cuts our work in half.

The next transitional value of σ is $\frac{1}{\sqrt{5}}$. In this case, the vertices of the small stellated dodecahedron of $\frac{1}{\sqrt{5}}\underline{D}$ are exactly the midpoints of the faces of $\underline{D}$. Thus, the icosahedron whose twelve vertices are the apices of the cells $\frac{1}{\sqrt{5}}\underline{D}_1$ is inscribed in the dodecahedron $\underline{D}$. Reciprocally, we see that $\sigma = \sqrt{5}$ is also a transitional value.

Next, we have $\sigma = \frac{1}{\tau} = (\sqrt{5}-1)/2$ as a transitional value. In this case, the "outermost" edges of the cells $\frac{1}{\tau}\underline{D}_2$ perpendicularly bisect the edges of $\underline{D}$. Said another way, the outer edges of the great dodecahedron of $\frac{1}{\tau}\underline{D}$ meet the edges of $\underline{D}$ at their respective midpoints and are orthogonal to each other. Reciprocally, we see that $\sigma = \tau$ is also a transitional value.

Finally, we see that $\sigma = 1$ is a transitional value for which the only nondegenerate cells are those represented by ii, with $0 \leq i \leq 7$.

III Cell Status

We shall not offer here a detailed analysis of the determination of cell status. Rather, we will present a suitable methodology for creating the tables in Figure 2. One may proceed as follows:

1. Determine the status of the cells when $\sigma = 0$. Using the same "reciprocity" discussed in (II) above, we find that the status of cell ij for $\sigma = 0$ is the same as the status of cell ji for $\sigma = \infty$. Thus, the status of cells when $\sigma = \infty$ is thereby determined.
2. Determine the status of the cells when $\sigma = 1$. As remarked above, ij is nondegenerate exactly when $i = j$. Moreover, ij is degenerate when cells D_i and D_j meet only at a vertex, edge, or face of a cell. This happens often, as seen in Figure 2.
3. Note that D_i is bounded when $0 \le i \le 3$. As a result, ij cannot be unbounded if $0 \le i \le 3$ or $0 \le j \le 3$. This further implies that such cells, if degenerate for two values of σ, must be bounded-nd for all values between these two. Likewise, if $0 \le i \le 3$ or $0 \le j \le 3$ and cell ij is bounded-nd for one value of σ and degenerate for another, then it must be bounded-nd for all values between these two.

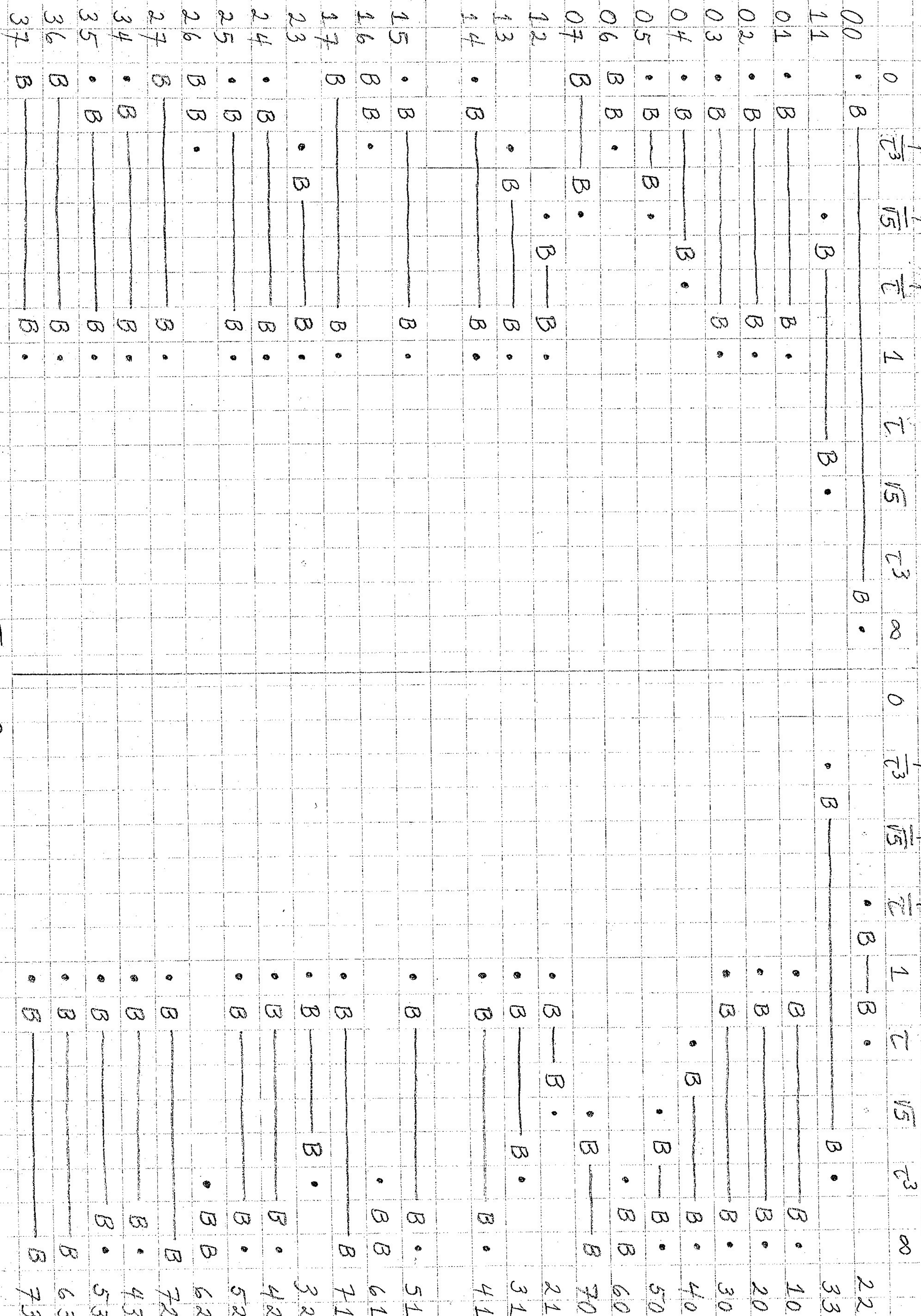

SCALE FACTORS, σ
0
$\frac{1}{\tau^3}$
$\frac{1}{\sqrt{5}}$
$\frac{1}{\tau}$
1
τ
$\sqrt{5}$
τ^3
∞
00 11 01 02 03 04 05 06 07 12 13 14 15 16 17 23 24 25 26 27 34 35 36 37
22 33 10 20 30 40 50 60 70 21 31 41 51 61 71 32 42 52 62 72 43 53 63 73

Figure 2

4. Complete the table, always keeping in mind that the status of cell ij at the scale factor σ is the same as the status of cell ji at the scale factor $\frac{1}{\sigma}$ (reciprocity again). Also, don't lose sight of the geometrical relationships by which the transitional values of σ were determined. Finally, it is very useful to have actual models or pictures of models of the stellations of the dodecahedron handy during your analysis. _Polyhedron Models_ would be a good choice of reference.

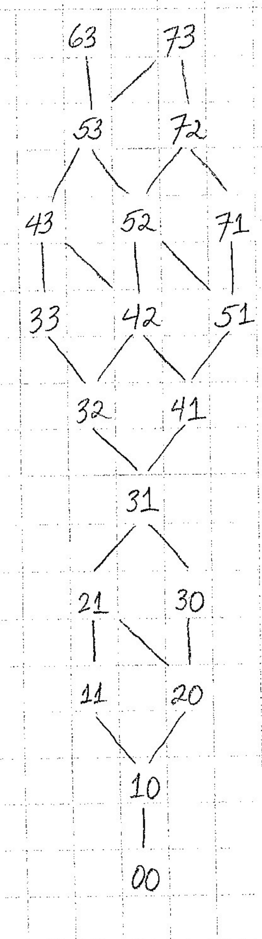
This analysis yields Figure 2. Note that the status of sixteen cells, namely 44, 45, 46, 47, 54, 55, 56, 57, 64, 65, 66, 67, 74, 75, 76, and 77, has been omitted since these cells are not bounded-nd for any value of σ.

IV. Cell Diagrams

The heuristics used for the creation of the cuboctahedral cell diagrams may also be used here. It is evident that the stellations in $S[\underline{D}, \sigma\underline{D}]$ are the same as those of $S[\underline{D}, \frac{1}{\sigma}\underline{D}] = S[\frac{1}{\sigma}\underline{D}, \underline{D}]$, those of $S[\underline{D}, \sigma\underline{D}]$ being precisely those of $S[\frac{1}{\sigma}\underline{D}, \underline{D}]$ scaled by a factor of σ. Consequently, the cell diagram for a particular value of σ must be the same as that for $\frac{1}{\sigma}$.

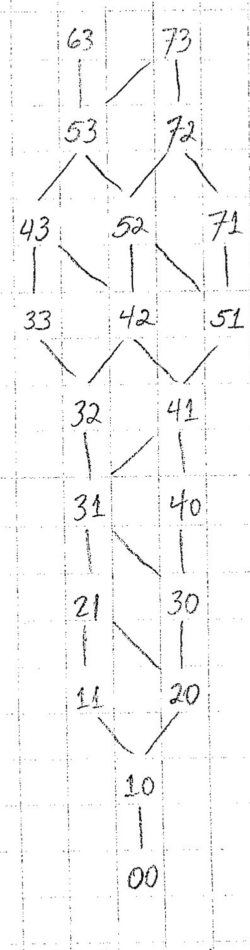
Indeed, it again is a result of reciprocity that to obtain the cell diagram for $\frac{1}{\sigma}$ from that for σ, one simply replaces each cell ij occurring in the diagram with the cell ji. Thus, there are essentially only nine (rather than seventeen) different cell diagrams corresponding to the values $0 \leq \sigma \leq 1$. By replacing "D_i" with "ii" in Figure 1, the cell diagram for $\sigma = 1$ is obtained.

Four other cell diagrams are included below, as they shall be referred to later. The reader may readily construct the others, if desired.



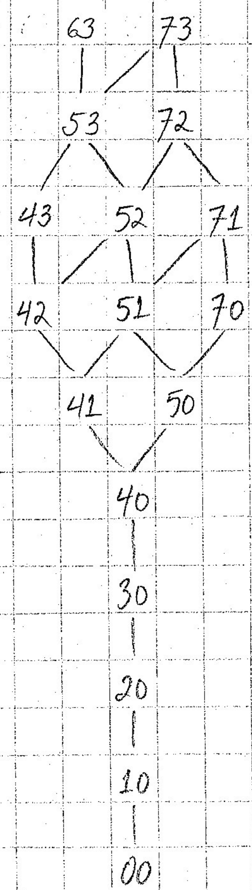
$\sigma = \tau$

Figure 3



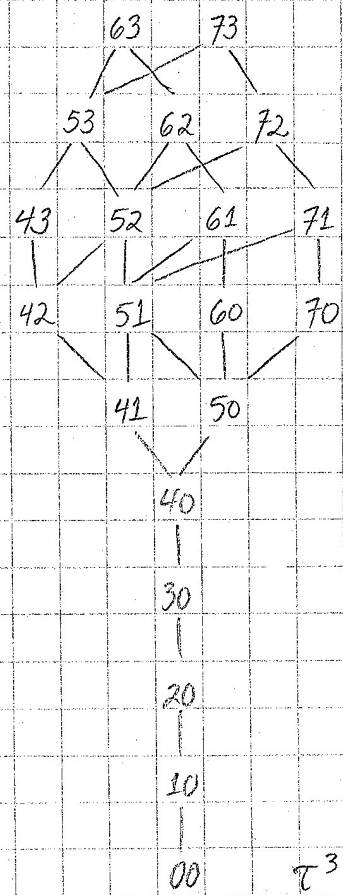
$\tau < \sigma < \sqrt{5}$

Figure 4



$\sigma = \tau^3$

Figure 5



$\tau^3 < \sigma < \infty$

Figure 6

One important difference in the construction of dodecadodecahedral cell diagrams regards the assignment of a level to a cell. Note that since both $\underline{D}_6$ and $\underline{D}_7$ are on top of $\underline{D}_5$, they both belong to the same level (in this case, level 6, as in Figure 1). Thus, if neither i nor j is seven, then the level of cell ij is $i+j$; otherwise, it is $i+j-1$. (Of course, the level of 77 is only 12, but this cell is always unbounded, and therefore will not occur in a cell diagram here.) For example, the level of cell 63 is 9, but so is the level of cell 73. This difference is noticeable in Figures 3–6.

V. STELLATIONS

We shall not, as in the cuboctahedral case, enumerate the complete set of fully-supported stellations corresponding to each value or range of σ. Indeed, Figure 3 alone would contribute 47 stellations to our list! As the technique is the same as for the cuboctahedral case, the industrious reader may enumerate at his or her leisure.

Our approach shall be more modest. As it happens, there are six uniform polyhedra which are dodecadodecahedral stellations for some value of σ. We enumerate these below. (Of course, the four dodecahedral stellations (<u>5</u>, <u>20</u>, <u>21</u>, and <u>22</u> in PM) may also be found among the dodecadodecahedral stellations, but they will be classified with regard to their simpler dodecahedral representation. In fact, one may easily convince oneself that these four are in $S[\underline{D}, \sigma\underline{D}]$ for <u>every</u> value of σ.)

1. THE DODECADODECAHEDRON (<u>73</u> in PM).

   This uniform polyhedron belongs to $FS[\underline{D}, \tau\underline{D}]$, and is in fact the fully-supported stellation $[\![20]\!]$ (see Figure 3). By reciprocity, we see that it is also the stellation $[\![02]\!]$ in $FS[\![\underline{D}, \frac{1}{\tau}\underline{D}]\!]$. In the future, such statements about reciprocity shall be left to the reader.

2. THE TRUNCATED GREAT DODECAHEDRON (<u>75</u> in PM).

This polyhedron belongs to $FS[\underline{D}, \sigma\underline{D}]$, where $\sigma = \frac{2}{\sqrt{5}}(1 + \sin\frac{1}{10})$ (here, angles are measured in revolutions, so that $\frac{1}{10} = \frac{\pi}{5}$ radians $= 36$ degrees). This polyhedron is also ⟦20⟧ (see Figure 4).

3. THE DITRIGONAL DODECAHEDRON (<u>80</u> in PM).

This uniform polyhedron is a member of $FS[\underline{D}, \tau^3\underline{D}]$. To see how the scale factor τ^3 was determined, consider Figure 7 (taken from p. 123 of PM). One readily sees by examining <u>80</u> that the two heavily outlined pentagons in Figure 7 are faces of the two dodecahedra $\underline{D}$ and $\sigma\underline{D}$ whose 24 facial planes contain the 24 faces of <u>80</u>; i.e, <u>80</u> belongs to $S[\underline{D}, \sigma\underline{D}]$, where $\sigma = \frac{d}{a}$. This ratio is easy to calculate, since the abundance of isosceles "$\frac{1}{10}$-$\frac{1}{5}$-$\frac{1}{5}$" triangles (that is, "36°-72°-72°" triangles) yields the relationships $b = \tau a$, $c = \tau b$, and $d = \tau c$, from which it is easily seen that $d = \tau^3 a$; i.e, $\frac{d}{a} = \tau^3$. Finally, we note that this

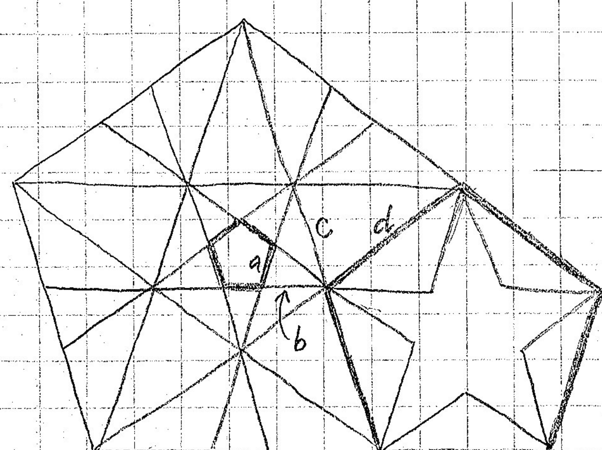


Figure 7

polyhedron is the stellation [00, 10, 20, 30, 50, 70] in $S[\underline{D}, \tau^3 \underline{D}]$ (see Figure 5). We remark that the calculations which determine the scale factors in the other examples may likewise be executed, but they are occasionally somewhat tedious and are left to the reader

4. THE QUASITRUNCATED SMALL STELLATED DODECAHEDRON ($\underline{97}$ in PM)

This polyhedron belongs to $FS[\underline{D}, \sqrt{5}\tau^3 \underline{D}]$ It may be considered as the dodecahedron $\sqrt{5}\tau^3 \underline{D}_0$ (which is the fully-supported stellation $[\![60, 70]\!]$) together with the cells $4\underline{1}$, 51, and 61; that is, it is the stellation $[\![61, 70]\!]$ (see Figure 6).

5. THE SMALL DODECAHEMIDODECAHEDRON ($\underline{91}$ in PM)

This uniform polyhedron belongs to $FS[\underline{D}, 0\underline{D}]$. We see that the dodecahedron D_0 is composed of the cells 06 and 07, and hence $\underline{91}$ is just the stellation [07].

6. THE GREAT DODECAHEMIDODECAHEDRON ($\underline{107}$ in PM)

This polyhedron, like the previous one, belongs to $FS[\underline{D}, 0\underline{D}]$. It is an "enlargement" of $\underline{91}$ in the following sense: if we enlarge the equatorial decagons of $\underline{91}$ to decagrams and enlarge the pentagons of $\underline{91}$ to pentagrams, we obtain $\underline{107}$. It is the fully-supported stellation $[\![07, 26]\!]$