5 May 99
(Revised from 13 Aug 91)

Creating Stellation Diagrams

We now look at the mathematics of constructing a stellation diagram for a given face $\mathcal{F}$ of a polyhedron $\underline{P}$. (see Figure 1(a)). Let $\mathcal{P}$ be the plane containing $\mathcal{F}$, and let b be the point in $\mathcal{P}$ such that the vector $\underline{b}$ from O (the barycenter of the polyhedron) to b is normal to $\mathcal{P}$.

Now suppose we desire to find the line in the stellation diagram for $\mathcal{F}$ determined by the face $\mathcal{F}_1$. If $\mathcal{P}_1$ is the plane containing $\mathcal{F}_1$, we simply seek the line ℓ which is the intersection of the planes $\mathcal{P}$ and $\mathcal{P}_1$.

In order to create a graphical representation of ℓ (such as might be done using a computer program), it is helpful to impose a coordinate system on $\mathcal{P}$. We choose b as our origin, and select an arbitrary vector $\underline{r}$, parallel to $\mathcal{P}$, as a "reference vector", which we may imagine as the direction of a "positive y-axis" in our diagram (See Figure 2.)

As is evident from examining Figure 2, if we determine the distance δ from b to ℓ and the angle θ which ℓ makes with the reference vector $\underline{r}$, we may suitably situate ℓ in $\mathcal{P}$, the facial plane of $\mathcal{F}$.

Finding δ is an exercise in Euclidean geometry. Referring to Figure 3, the industrious reader could no doubt conclude that

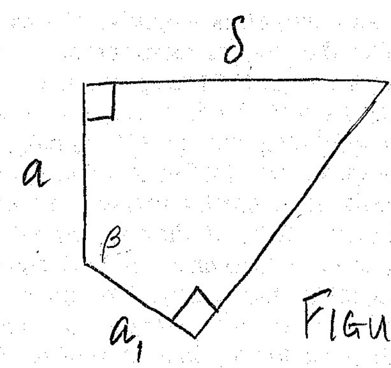


FIGURE 3

$$\delta = \frac{a_1 - a\cos\beta}{\sin\beta} \qquad (1)$$

(This formula is valid for both acute and obtuse θ.)

In our case (see Figure 1(b)), we have

$$a = \|\underline{b}\|, \quad a_1 = \|\underline{b}_1\|, \quad \cos\beta = \frac{\underline{b}\cdot\underline{b}_1}{\|\underline{b}\|\,\|\underline{b}_1\|}; \qquad (2)$$

substituting these values in (1) yields

$$\delta = \frac{\|\underline{b}_1\| - \|\underline{b}\|\cos\beta}{\sin\beta} = \frac{\|\underline{b}\|\left(\|\underline{b}_1\|^2 - \underline{b}\cdot\underline{b}_1\right)}{\sqrt{\|\underline{b}\|^2\|\underline{b}_1\|^2 - (\underline{b}_1\cdot\underline{b}_2)^2}} \qquad (3)$$

When our polyhedron $\underline{P}$ is such that all faces are the same distance from the barycenter of $\underline{P}$, we must have $\|\underline{b}\| = \|\underline{b}_1\|$, so that (3) becomes

$$\delta = \|\underline{b}\| \frac{1-\cos\beta}{\sin\beta} = \|\underline{b}\| \tan\frac{\beta}{2}. \tag{4}$$

It is an easy matter to find θ; we simply have

$$\cos\theta = \frac{\underline{r}\cdot(\underline{b}\times\underline{b}_1)}{\|\underline{r}\|\,\|\underline{b}\times\underline{b}_1\|} \tag{5}$$

With the use of (3) and (5), it is possible to write a short computer program which produces a stellation diagram given the face normals (such as $\underline{b}$) for our polyhedron $\underline{P}$. If $\underline{P}$ is the dual of an Archimedean solid $\underline{A}$, then the face normals for $\underline{P}$ may be described by vectors from the barycenter of $\underline{A}$ to the vertices of $\underline{A}$. For some of the simpler duals (such as the triakis tetrahedron), these calculations may reasonably be done by hand.

For completeness, we briefly remark on the determination of $\mathbf{b}$ (see Figure 4). If b in the facial plane of $\mathcal{F}$ is such that the vector $\mathbf{b}$ from 0 to b is normal to the facial plane $\mathcal{P}$, we may find $\mathbf{b}$ as follows. Let $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ be three vectors from 0 to vertices of $\mathcal{F}$, and put

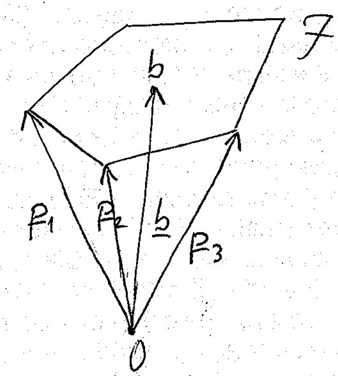


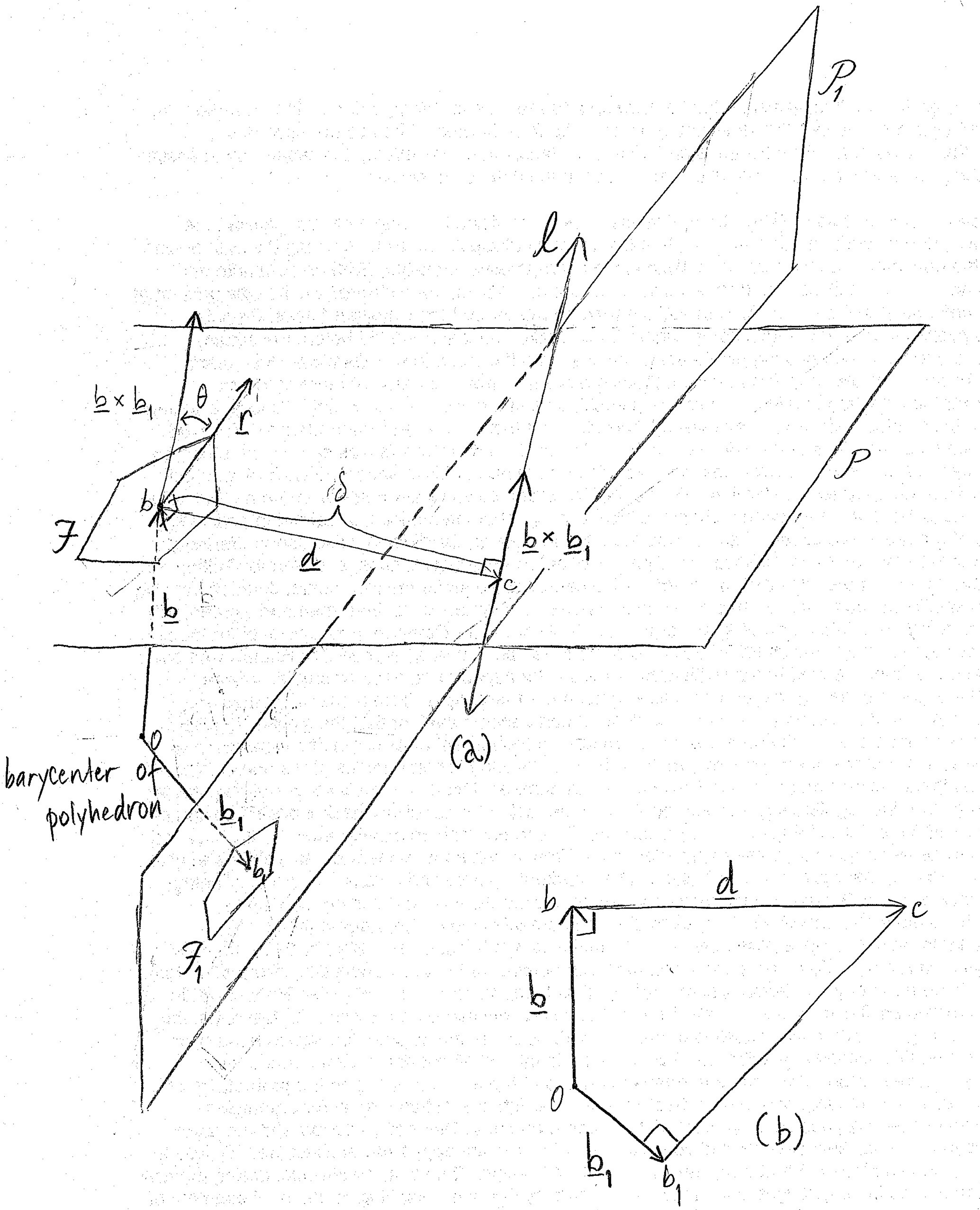
FIGURE 4

$$\mathbf{n} := (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1).$$

A routine calculation gives us

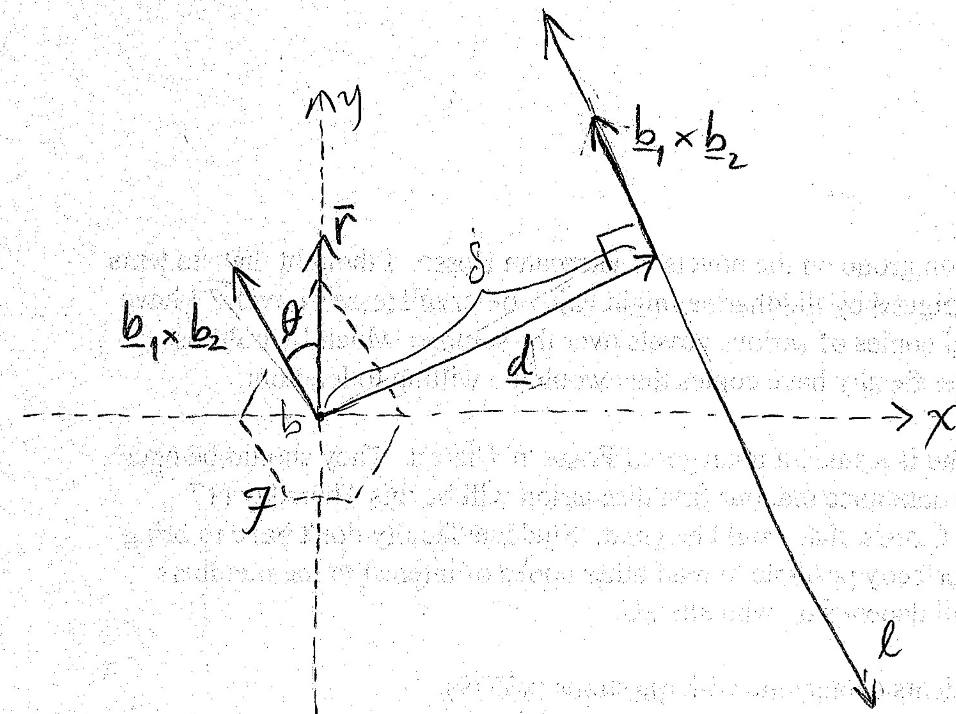
$$\mathbf{b} = \frac{(\mathbf{n} \cdot \mathbf{p}_1)\,\mathbf{n}}{\mathbf{n} \cdot \mathbf{n}} \tag{6}$$

As usual, any permutation of the subscripts "1", "2", and "3" yields the same result.



Two-dimensional cross-section of $\underline{b}$, $\underline{b}_1$, and $\underline{d}$

FIGURE 1.



y
$\underline{b}_1 \times \underline{b}_2$
$\bar{r}$
δ
$\underline{b}_1 \times \underline{b}_2$
θ
$\underline{d}$
b
x
7
ℓ

FIGURE 2.