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Challenging Gifted High School Students

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ABSTRACT: *At Quincy Senior High School in Quincy, Illinois, USA, an increasing number of eleventh graders complete calculus. The Creative Problem Solving in Mathematics course (CPSM) was designed to challenge these students in the twelfth grade. A brief history of the development of the course will be followed by a week-by-week syllabus of topics covered. Several illustrative examples of course content will then be given.*

INTRODUCTION AND MOTIVATION

In the United States, gifted mathematics students often take calculus in the eleventh grade. In the twelfth grade, many students take an Advanced Placement course in statistics or perhaps enroll in a further semester of calculus at a local community college.

Is this the best way to encourage our best and brightest mathematics students? What is missing in this approach?

Before answering this question, it is helpful to review a few recent reforms in mathematics education in the United States. These reforms, in part, helped create educational environments wherein students routinely complete calculus before graduating from high school.

In 1980, the National Council of Teachers of Mathematics (NCTM) published *An Agenda for Action: Recommendations for School Mathematics of the 1980's.* In this monograph, the NCTM recommended that "[m]ore mathematics should be required for all students and a flexible curriculum with a greater range of options be designed to accommodate the diverse needs of the student population." As this recommendation, among others, began being implemented in schools, the NCTM perceived a need to develop the *Curriculum and Evaluation Standards for School Mathematics.* Published in 1989, this document called for a common mathematical experience to be delivered to all students in the United States. This experience was to enhance opportunities to learn solid mathematics and help prepare every student for entry into the work force, college-bound or not.

Around this time, the AMOCO Foundation funded the University of Chicago School Mathematics Project which had two parallel development phases. The K–6 (kindergarten through sixth grade) phase examined international texts and research on child development in order to develop a mathematically rich curriculum which challenges all students and encourages mathematical growth through experiences related to their daily lives.

The secondary phase (seventh grade through twelfth grade) had as its "...most fundamental feature, its focus on upgrading the mathematics experience of the average student" [Usiskin, 1990, p. 4]. It provided stimulating applications of mathematics and introduced topics not usually presented to average or even gifted students until college. For example, matrix algebra and transformations, and linear programming are included in the UCSMP *Advanced Algebra* text used by average eleventh grade students.

Beyond these reforms, the authors have had varied experience with coaching mathematics teams for the Illinois Council of Teachers of Mathematics (ICTM) Regional and State Mathematics Contests. One strong feature of the ICTM Mathematics Contests is the Oral Competition. Students participating in this competition study a topic (different each year) which is not normally taught in a high school curriculum, such as geometrical inversion or error-correcting codes. Often such topics are not even included in a standard undergraduate mathematics curriculum.

During the competition, each student is given fifteen minutes to prepare solutions to three problems related to that year's topic, and seven minutes to present their solutions to two university professors who score the presentations based on a common rubric. The professors may then ask follow-up or clarifying questions for three minutes after the presentation.

Preparation for the Oral Competition also included a simulation of contest conditions where a student would present solutions to his or her peers on the mathematics team. Based on observations of both the oral competitor and the nature of the follow-up questions by peers, it was evident that these non-traditional topics were of significant interest to the mathematics team students and helped motivate the team.

Such experience with mathematics competitions, together with an ongoing relationship with an outstanding mathematics community in Illinois (notably John Benson, John Dossey, John McConnell, and Zalman Usiskin), suggests an answer to the question posed earlier. What is missing from the mathematics education of our brightest high school students is a significant exposure to non-traditional topics. This includes having students read, reason, and write about mathematics at a college level, focus intensely on problem-solving, and apply their knowledge to practical situations.

Of course this question is not answered in a vacuum. One of the social issues involved in Quincy, Illinois is the loss of the brightest high school graduates. Although Quincy is an urban environment with strong medical, electronics, and arts communities, a rich architectural history, a four-year private university, and a two-year community college, its geographic remoteness is not attractive to young college graduates. Most of Quincy Senior High School's top students do not return home after college. Offering our gifted students a stimulating non-traditional mathematical experience may strengthen their ties to Quincy.

Introducing a course like CPSM also has an impact in the educational environment. If students are expected to read, write, and think about mathematics at an advanced level in their capstone course, teachers begin to see it as their task to prepare students for such a course. This begins to change the way teachers themselves teach and think about mathematics and the coursework they deliver.

The capstone course also helps guide instruction. For example, the construction of Platonic solids and other polyhedra now occurs in other grades. In 2004, CPSM students and instructors went to several sixth grade classrooms and helped them build various types of dodecahedra. In general, educators in Quincy schools tend to be more familiar with polyhedra than they were a decade ago. Thus, upgrading the mathematics experience of the best students and their teachers supports the upgrading of the mathematics experience of average students and their teachers.

THE DEVELOPMENT OF CPSM

Development of CPSM began in 1996. A wide variety of stakeholders were involved in the planning: mathematics team students who were enrolled in calculus as eleventh graders, former students who were in the gifted program at Quincy Senior High School, parents of gifted students, key members of the electronics research and medical communities, mathematics and science faculty members at Quincy Senior High School, and a mathematics faculty member from Quincy University. Over the course of a month, the team grappled with designing a course which best challenged and served the needs of the brightest students while meeting the community's need to encourage students to return to Quincy after graduating from college.

The need to round out the mathematics experience of traditionally college-bound students, the excitement generated by the topics of the Oral Competition, and the students who assisted in the course design strongly influenced the structure and content of the course. The outcomes of CPSM agreed upon were:

1. The student will explore mathematics topics beyond calculus.
2. The student will conduct a research project, with the assistance of a mentor from the community, if possible.
3. The student will use project management and teamwork skills in conducting and evaluating the research project.

COURSE HISTORY

The first Creative Problem Solving in Mathematics course was run during the 1997–1998 academic year. Four students enrolled, and were taught by Todd Klauser of Quincy Senior High School and Vince Matsko of Quincy University. Mathematical topics included were taxicab geometry, geometrical inversion, polyhedra, finite differences, mathematical envelopes, probability, number theory, and spherical trigonometry. The class designed and built a four-frequency icosahedron, which established a tradition of doing class projects. One day a week was set aside as Problem Day, where students worked on problems from a variety of mathematical areas. Each student also presented the results of an individual research project.

During the second year, enrollment increased to eleven students, including one student who took the course through independent study due to scheduling conflicts. As enrollment grew, the instructors grew more comfortable with course content and flow, which allowed for altering the content from time to time. In addition, a larger number of students allowed for more creative and extensive class projects.

Now enrollment is typically 12–15 students (Quincy Senior High School has approximately 1700 students in grades 10–12, with 36 juniors in calculus). Currently, the course is based on an innovative geometry manuscript, *Polyhedra and*

Geodesic Structures [Matsko, 2005]. A detailed outline of the course is provided below. Former students occasionally return as guest speakers to talk about careers in mathematics-related fields.

Students have found the course exciting and valuable. One former student remarked that CPSM was the most beneficial course in preparing for college. Another, referring to her individual project, said, "it has been the most wonderful experience for me." Others said, "I [was] challenged and learn[ed] about new areas of math that I never new existed," and "Because of the class size we [were] able to enjoy the learning in an environment unlike that of any of my other classes."

SYLLABUS

Below is the current topic-by-topic syllabus of CPSM, with the approximate length of time spent on each topic. Class meets five days a week for 47 minutes. One day every two weeks is Problem Day, which consists of presentations of solutions to two problems assigned over the two-week period. These problems are chosen to expose students to various topics in mathematics and to develop technical writing ability. Problem areas include number theory and Diophantine equations, combinatorics, calculus, and probability.

Students select individual topics for a research project sometime in the third quarter. Occasional class days are devoted to work on these projects. The three-week project period at the end of the year allows for students to give twenty-minute presentations on their projects. They must also write a ten-page summary paper. Students sometime work in pairs on larger projects. Past projects include: building stellations of an irregular dodecahedron, constructing a harmonograph, writing programs to render three-dimensional computer graphics, cryptography, and designing a geodesic house.

In addition to the individual projects, students undertake a more extensive class project. For a recent example, see the Q-Ball at http://www.vincematsko.com/.

Hands-on work, whether in the form of drawing mathematical envelopes or building polyhedra, is an integral part of the course. A "$\star$" next to a topic indicates that individual or class building projects are a part of that unit. A "$\diamond$" next to a topic indicates that students use Geometer's Sketchpad during the unit.

Chapter numbers refer to the draft manuscript, *Polyhedra and Geodesic Structures* [Matsko, 2005]. Some chapters are not covered in class but are handed out for selfstudy and possible use for individual projects. Material for other topics is given as class notes.

Some of these topics are illustrated below by means of detailed examples. When this is the case, “see below” is included in the description.

1. $\diamond$ Basic Constructions (Appendix A, 1 week). Basic compass and straightedge constructions are reviewed.

2. Trigonometry (Chapter 0, 1 week). A review of important trigonometric relationships is given.

3. $\diamond$ Angles and Constructions (Chapter 1, 1 week, see below). The construction of regular figures and the trigonometric functions of $36°$ and $72°$ angles are introduced.

4. $\star$ $\diamond$ The Platonic Solids (Chapter 2, 1 week, see below). Geometric and algebraic enumerations of the Platonic solids are given.

5. $\star$ $\diamond$ Spherical Trigonometry (Chapter 3, 2 weeks, see below). Basic formulas are derived and used to calculate the edge and dihedral angles of the Platonic solids. Non-Euclidean considerations are emphasized.

6. $\star$ $\diamond$ Taxicab Geometry (2 weeks, see below). Students explore the geometry of the “taxicab” metric defined by

$$d((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|.$$

7. $\star$ $\diamond$ Geodesic Structures (Chapter 4, 2 weeks). Spherical trigonometry is applied to the design and construction of geodesic spheres.

8. $\star$ $\diamond$ The Archimedean Solids (Chapter 5, 1 week). The Archimedean solids are enumerated both geometrically and algebraically.

9. Angles and Archimedeans (Chapter 6, 2 weeks). Spherical trigonometry is applied to calculating the edge and dihedral angles of the Archimedean solids.

10. $\diamond$ Geometrical Inversion (2 weeks). Inversion in a circle is presented, including extending the plane by adding a point at infinity.

11. $\star$ $\diamond$ Geodesic Structures, II (Chapter 7, 1 week). Further techniques for creating geodesic spheres are derived using spherical trigonometry.

12. Antiprisms and Snub Polyhedra (Chapter 8, handout only).

13. $\star$ Duality (Chapter 9, 1 week). Duals of the Platonic and Archimedean solids are discussed. Their edge and dihedral angles are calculated using spherical trigonometry.

14. Geodesic Structures, III (Chapter 10, handout only).

15. $\star$ Deltahedra (Chapter 11, 1 week). The deltahedra – convex polyhedra with equilateral triangular faces – are enumerated. Dihedral angles are found using spherical trigonometry.

16. Kepler-Poinsot Polyhedra (Chapter 12, handout only).

17. Euler's Formula (Chapter 13, handout only).

18. Coordinates of Polyhedra (Chapter 14, 2 weeks, see below). Cartesian coordinates in three dimensions are found for the vertices of the Platonic solids and some Archimedean solids.

19. $\diamond$ Mathematical Envelopes (2 weeks, see below). A parameterized family of lines gives the illusion of curvature; the apparent curve is the *envelope.* Calculus is used to find a Cartesian equation for an envelope given a parameterization of lines.

20. Matrices and Symmetry Groups (Chapter 15, 2 weeks). The symmetry groups of some of the Platonic solids are represented as groups of matrices.

21. Graph Theory and Polyhedra (Chapter 16, 2 weeks). The adjacency of vertices on a polyhedron may be represented as a graph. Various properties of such graphs are discussed.

22. Projects (3 weeks).

Of course a class syllabus is necessary in order for educators to evaluate whether a course such as CPSM might fit into their curriculum. It is equally important, however, to give a feel for the nature of the mathematics presented and the level of rigor used in the classroom. Several detailed examples are included to illustrate the variety of mathematics covered and the way that topics are presented.

ANGLES AND CONSTRUCTIONS

The usual constructions with straightedge and compass are reviewed, such as bisecting an angle, dropping a perpendicular to a line from a point, or trisecting a segment. In addition, students construct regular pentagons, decagons, and pentadecagons (15-gons).

The golden ratio τ (sometimes called ϕ in algebra) occurs frequently in a study of three-dimensional polyhedra involving pentagons. For the planar case, consider the pentagon in Figure 1. It is clear that Δpqr and Δqst are similar isosceles triangles with apex angle $36°$ and base angles of $72°$. Call the ratio of the length of the longer sides to that of the shorter side of either triangle ρ. Let the notation "$[pq]$" denote the length of the segment pq. With $x = [pq] = [qr] = [qt]$ and $y = [pr] = [rs]$, consideration of these similar triangles shows that

$$\rho = \frac{x}{y} = \frac{x+y}{x} = 1 + \frac{y}{x} = 1 + \frac{1}{\rho}.$$

Multiplying through by ρ yields $\rho^2 = \rho + 1$, and hence we obtain the quadratic equation $\rho^2 - \rho - 1 = 0$. By applying the usual quadratic formula, we see that this equation has one positive and one negative root. Since our ratio is positive, we choose the positive root, resulting in $\tau = (1 + \sqrt{5})/2$.

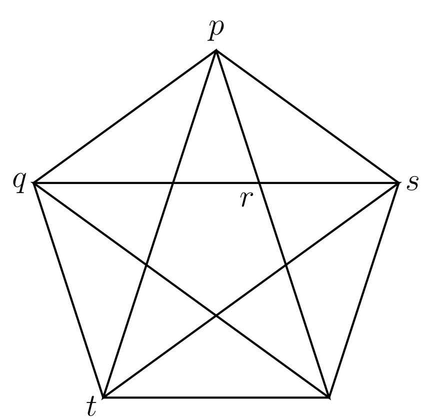


Figure 1

We may use these geometric facts in trigonometry as well. By considering the isosceles triangles mentioned above, it follows that

$$\sin 18° = \cos 72° = \frac{1}{2}(\tau - 1), \quad \sin 54° = \cos 36° = \frac{\tau}{2}.$$

THE PLATONIC SOLIDS

A *Platonic solid* is a convex polyhedron whose faces are all the same regular polygon, with the same number of polygons meeting at each vertex. It is well-known that there are five Platonic solids.

To see this algebraically, let p denote the number of sides on each face of a Platonic solid $\mathcal{P}$, and let q denote the number of faces meeting at each vertex. Also, let V, E, and F denote the number of vertices (corners), edges, and faces, respectively on $\mathcal{P}$. A standard result for convex polyhedra is Euler's formula:

$$V - E + F = 2.$$

This may be illustrated with the example of cube, which has 8 vertices, 12 edges, and 6 faces, with $8 - 12 + 6 = 2$.

Since there are p edges on each of F faces, there are pF edges on the faces of $\mathcal{P}$. But this counts each edge of $\mathcal{P}$ twice, so that $pF = 2E$. A similar argument yields $qV = 2E$. Solving these relationships for E and substituting back into Euler's formula yields

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.$$

This is an example of a *Diophantine equation;* that is, an equation with integer solutions. Since regular polygons have at least 3 sides and at least 3 polygons meet at each vertex of a convex polyhedron, p and q must be integers 3 or greater. Note that if both $p \geq 4$ and $q \geq 4$, then $1/p + 1/q \leq 1/2$, so that at least one of p and q must be 3. This allows all solutions to be enumerated:

Platonic solid	p	q	V	E	F
Tetrahedron	3	3	4	6	4
Cube	4	3	8	12	6
Octahedron	3	4	6	12	8
Dodecahedron	5	3	20	30	12
Icosahedron	3	5	12	30	20

Table 1

The Platonic solids may also be enumerated geometrically.

A *net* for a polyhedron is an arrangement of polygons in the plane which may be cut out and folded to make a three-dimensional model of the polyhedron. Students design their own nets for the Platonic solids using Geometer's Sketchpad, and then use these nets to construct models using heavy paper, scissors, and glue. (Nets are also included in the text.)

Students are also guided through a set of exercises which algebraically enumerate the Archimedean solids. This enumeration is quite a bit more involved than the Platonic case.

SPHERICAL TRIGONOMETRY

A *spherical triangle* is a triangle on the surface of a sphere, the sides of which are arcs of great circles of the sphere. We find examples of great circles on a sphere by considering lines of longitude and the equator on a spherical globe. We may trace out a spherical triangle on a globe by beginning at the North Pole, following $0°$ of longitude to the equator, travelling west until we hit $60°$ of longitude, and following $60°$ of west longitude north back to the North Pole (see Figure 2).

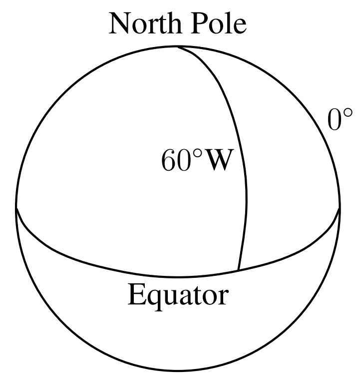


Figure 2

In the plane, we measure the six parts of a triangle by measuring the lengths of the sides and the measures of the angles between adjacent sides. The situation is somewhat different for spherical triangles, since all parts are angles.

Recall that each side of a spherical triangle is an arc of a great circle – and thus can be measured in degrees relative to that great circle (which has the same radius as the sphere). The angles between adjacent sides of a spherical triangle are called *vertex angles*. The vertex angle between two sides of a spherical triangle is just the angle between the two planes containing the great circles of which the sides are arcs. In the example above, the sides of the spherical triangle have measures $60°$, $90°$, and $90°$, and the three vertex angles have measures $60°$, $90°$, and $90°$ as well. (This coincidence is an accident of this particular example, and does not occur for every spherical triangle.)

We are used to the angles of a plane triangle adding to $180°$ regardless of the shape of the triangle. The measures of the vertex angles of the spherical triangle described above sum to $90° + 90° + 60° > 180°$. In fact, the sum of the vertex angles of a spherical triangle is always greater than $180°$, so that spherical geometry is an example of a non-Euclidean geometry. If we denote this sum by Σ, we find that $\frac{1}{720°}(\Sigma - 180°)$ is the fraction of the surface of the sphere occupied by the spherical triangle. Thus, our triangle occupies $\frac{1}{720°}(240° - 180°) = \frac{1}{12}$ of the surface of the sphere.

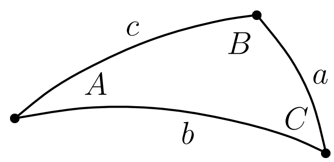


Figure 3

Of course formulas of plane trigonometry are not applicable to spherical triangles. But they may be used to derive some useful formulas for spherical trigonometry (see Figure 3):

$$\cos c = \cos a \cos b + \sin a \sin b \cos C, \tag{1}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{2}$$

As an example of the use of these formulas, consider the spherical triangle shown in Figure 4.

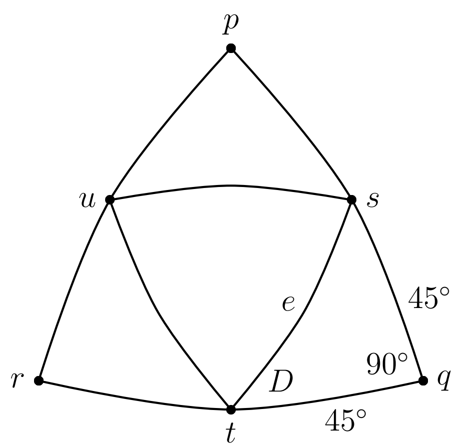


Figure 4

On a globe, p would be the North Pole, and q and r would be two points separated by $90°$ on the equator. All sides have measure $90°$, and all vertex angles also measure $90°$. Thus, Δpqr is both equilateral and equiangular.

Let s, t, and u be the midpoints of the sides of Δpqr, and join pairs of these points by arcs of great circles. In the Euclidean plane, joining midpoints of the sides of an equilateral triangle produces four smaller equilateral triangles.

What happens in the spherical case? Applying (1) to Δsqt results in

$$\cos e = \cos 45° \cos 45° + \sin 45° \sin 45° \cos 90°,$$

so that $\cos e = \frac{1}{2}$. Thus $e = 60°$, and hence Δsqt is *not* equilateral, although due to symmetry, Δstu is.

In the Euclidean case, D would have measure $60°$. Here, using (1) again, we see that

$$\cos 45° = \cos 45° \cos e + \sin 45° \sin e \cos D,$$

so that $\cos D = 1/\sqrt{3}$. Thus $D \approx 54.736°$.

This example also illustrates that equilateral triangles come in different sizes; equilateral Δpqr has sides measuring $90°$, while equilateral Δstu has sides measuring $60°$.

It takes students some time to be able to think "spherically" since most of what they know in Euclidean geometry does not carry over into spherical geometry. Study of spherical geometry is often the students' first exposure to a non-Euclidean geometry.

Spherical trigonometry is used extensively in the course to study the edge and dihedral angles of polyhedra and the design of geodesic spheres. As an interesting application, students create and build geodesic spheres by first performing the necessary calculations involving spherical trigonometry. With the help of the instructors, they then use these calculations to create the pieces needed to assemble the models.

TAXICAB GEOMETRY

Students are familiar with the properties of geometric figures in Euclidean geometry. These properties are based on a coherent system of postulates, theorems, and definitions. One such definition is that of the distance between two points, given by the usual formula

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

for points (x_1, y_1) and (x_2, y_2) in the plane.

In taxicab geometry, the definition of the distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$d_T((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$$

(see [Krause, 1986, p. 4]). This results in a non-Euclidean geometry which students can explore with a little guidance.

Consider the points $A(2, 2)$ and $B(8, 4)$, as shown in Figure 5. In Euclidean geometry, the midpoint of the segment between these two points may be defined in terms of the distance function as follows: the midpoint is that point M such that $d(A, M) + d(M, B) = d(A, B)$ and $d(A, M) = d(M, B)$. This point is uniquely determined, and is given by $M(5, 3)$. Note that the condition $d(A, M) + d(M, B) = d(A, B)$ ensures that M is on the line segment between A and B.

Consider the same definition of the midpoint using the taxicab distance. Note that the point $P(6, 2)$ satisfies $d_T(A, P) + d_T(P, B) = d_T(A, B)$ and $d_T(A, P) = d_T(P, B)$. In fact, $d_T(A, R) + d_T(R, B) = d_T(A, B)$ and $d_T(A, R) = d_T(R, B)$ for each point R on the line segment from P to Q. Thus, the midpoint of a segment cannot be determined uniquely on the basis of the distance function alone. When the additional assumption is made that the midpoint of a segment lies on that segment, then the midpoint may be uniquely determined in taxicab geometry. Recall that this additional assumption was not necessary in Euclidean geometry.

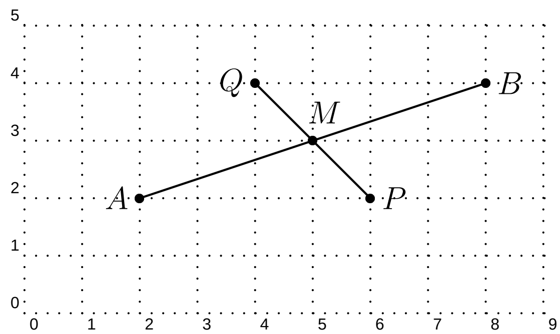
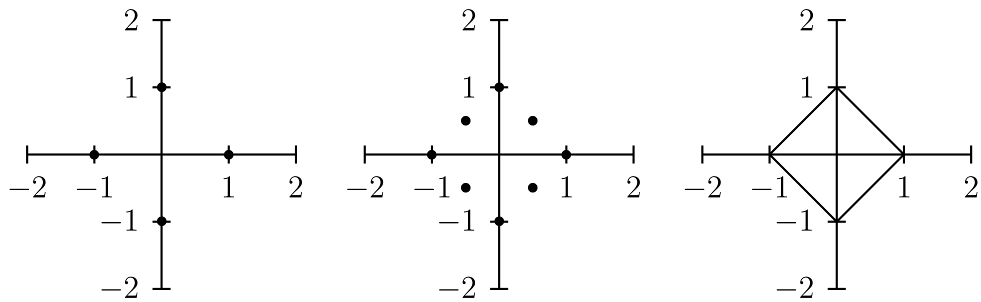


Figure 5

In Euclidean geometry, the figure defined by all points P with $d(P,(0,0)) = 1$ is usually referred to as the unit circle. What figure is determined by points P in taxicab geometry with the property $d_T(P,(0,0)) = 1$?

It is clear that moving one unit along either axis results in a point on this figure (see Figure 6(a)). However, the points $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},-\frac{1}{2})$, $(-\frac{1}{2},-\frac{1}{2})$, and $(-\frac{1}{2},\frac{1}{2})$ are also on this figure, as shown in Figure 6(b). When all possible points are included, the result is as shown in Figure 6(c). Thus the unit "circle" in taxicab geometry is in fact diamond-shaped.



(a) (b) (c)

Figure 6

If the symbol "π" refers to the ratio of the circumference of a circle to its diameter, it happens that this ratio is different in taxicab geometry from the usual value in Euclidean geometry. The diamond-shaped "circle" in Figure 6(c) has a circumference of 8 and a diameter of 2, so that $\pi = 4$ in taxicab geometry.

In addition to these geometrical considerations, there is a wealth of algebraic ideas to explore in taxicab geometry. As the distance from the point $P(x, y)$ to the point

$(0,0)$ is given by

$$d_T((x,y),(0,0)) = |x| + |y|$$

in taxicab geometry, it is evident that the diamond in Figure 6(c) may be found be graphing the Cartesian equation

$$|x| + |y| = 1.$$

Many interesting geometrical results in taxicab geometry may be proved or verified by such algebraic methods.

COORDINATES OF POLYHEDRA

For many applications, such as computer graphics or engineering problems, it is necessary to represent a problem in a three-dimensional coordinate system. To graphically represent a polyhedron in three dimensions, it is necessary to know the coordinates of its vertices.

The usual conventions for drawing a two-dimensional Cartesian coordinate system are to have the positive x-axis pointing due east and the positive y-axis pointing due north so that the axes are perpendicular. In creating a three-dimensional coordinate system, the third axis, called the *z-axis,* is imagined to point "straight up" so that it is perpendicular to the other two axes, just as walls in the corner of a room meet in mutually perpendicular line segments. The difficulty lies in trying to draw a picture of a corner – there is not enough room on a two-dimensional piece of paper for three mutually perpendicular axes.

There are a few different conventions for drawing a three-dimensional coordinate system in the plane. We will use the convention illustrated in Figure 7, where the y-axis points due east, the z-axis points due north, and the x-axis "comes out" in a southwesterly direction. When plotting a point in this coordinate system, it is easiest to first move along the x-axis according to the x-coordinate of the point, and then from this location, move according to the y- and z-coordinates as in two dimensions.

Our first task is to find the coordinates of the vertices of a cube. Recall that in two-dimensions, $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$ are vertices of a square with edge length 2, and the center of this square is $(0,0)$.

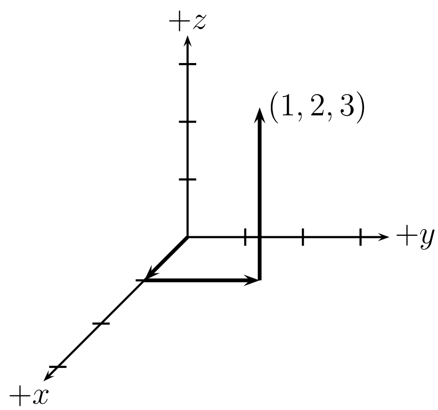


Figure 7

It is now an easy task to make a cube. Thinking of our square as lying in the xy-plane, moving it "up" one unit produces the square with vertices $(1, 1, 1)$, $(-1, 1, 1)$, $(-1, -1, 1)$, and $(1, -1, 1)$ in the plane $z = 1$. Moving the square "down" one unit results in the square with vertices $(1, 1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$, and $(1, -1, -1)$ in the plane $z = -1$. These translated squares are the top and bottom faces of the cube, as shown in Figure 8.

Note that the vertices of the cube consist of all lists of three coordinates, each of which is either $+1$ or -1. Since there are two choices for each of three coordinates, there are $2^3 = 8$ vertices on a cube. Also note that two vertices are adjacent if and only if they differ in exactly one coordinate; this coordinate indicates the axis to which the edge joining these two vertices is parallel.

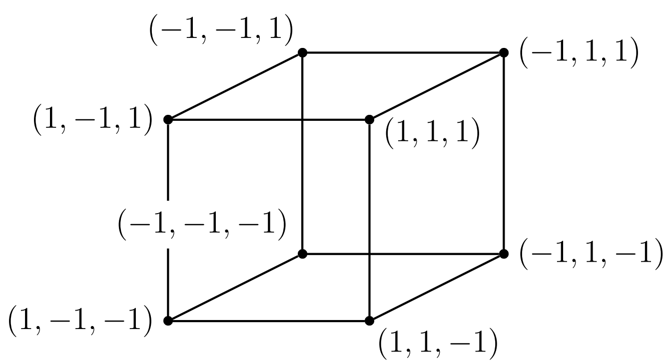


Figure 8

The midpoints of the edges of a cube are the vertices of a *cuboctahedron,* one of the

Archimedean solids (see Figure 9). Since coordinates for the vertices of the cube are known, coordinates for the vertices of the cuboctahedron may be found using the usual midpoint formula. For example, coordinates for the vertex P are given by

$$\left(\frac{-1-1}{2}, \frac{1+1}{2}, \frac{1-1}{2}\right) = (-1, 1, 0).$$

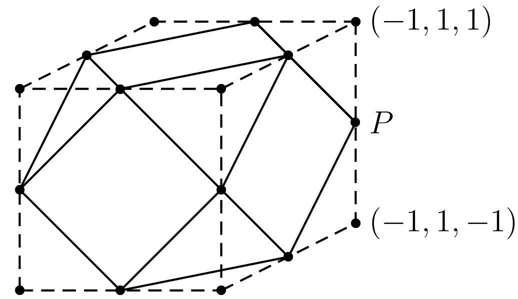


Figure 9

It is not hard to show that one of the coordinates of each vertex of the cuboctahedron is 0. In the example above, the z-coordinate is 0 because P is the midpoint of an edge of the cube parallel to the z-axis. The other two coordinates are either $+1$ or -1.

By relating other Platonic and Archimedean solids to the cube, coordinates for their vertices may also be found.

MATHEMATICAL ENVELOPES

A popular activity for students is creating geometrical figures from string, such as the one shown in Figure 10. This is an example of a *mathematical envelope of lines.* In other words, if enough lines are drawn tangent to a given curve, the illusion of the curve is produced. The curious student asks the natural question: what curve do we see?

Students can answer this question with a little calculus, as described in [Boltyanskii, 1964, p. 52]. To see how, consider Figure 10.

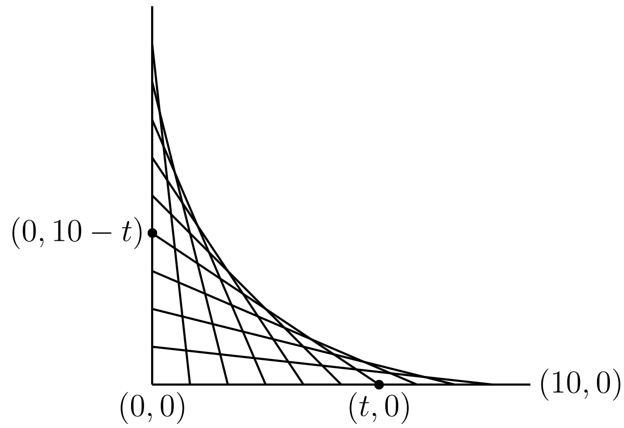


Figure 10

Using a convenient coordinate system, we might describe this envelope by saying it consists of lines in the first quadrant such that the x-intercept and y-intercept of each line sum to 10. Thus, if $(t, 0)$ is the x-intercept of such a line, then $(0, 10-t)$ is the y-intercept. An equation for the line determined by these two points is

$$y = -\frac{10-t}{t}x + 10 - t.$$

In order to make the next step easier, this equation may be rewritten as

$$(10-t)x + ty = t(10-t). \tag{3}$$

Each value of t produces a different line, so that t can be viewed as a parameter for the family of lines.

Now differentiate (3) with respect to t, resulting in

$$-x + y = 10 - 2t.$$

Solve this equation for t: $t = \frac{1}{2}(10 + x - y)$. Then substitute this expression for t back into (3) and rearrange terms:

$$x^2 - 2xy + y^2 - 20x - 20y + 100 = 0.$$

Using the usual test for characterizing a conic section, we find that this curve is a parabola.

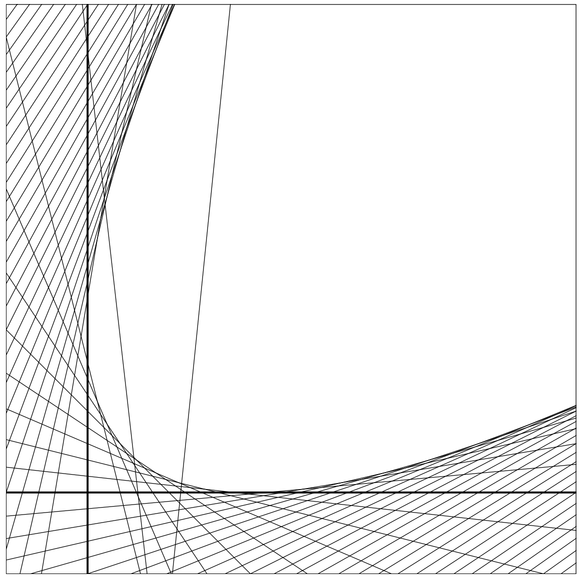


Figure 11

Even more dramatic is what happens when the lines in Figure 10 are extended beyond the coordinate axes, and values of t in (3) are allowed to be negative (as shown in Figure 11).

With guidance, our students tackle problems involving a significant amount of algebra. Using a ruler, pencil, and graph paper, or a program like Geometer's Sketchpad, they confirm that the result they derive using calculus corresponds to the envelope they created.

CONCLUDING REMARKS

Creative Problem Solving in Mathematics is a stimulating, challenging course for talented students. Its demands on the instructors are perhaps greater than for a typical high school course; the teacher may need to learn new topics and devise ways to present them at an appropriate level. In our case, the involvement of a university faculty member (Vince Matsko) as mentor to a high school teacher (Todd Klauser) was especially valuable. Currently, Matsko visits the CPSM classroom once or twice weekly as time permits. It is important to consider either release time from the usual course load or other form of compensation when involving a university faculty mentor.

Also crucial is the support of school administrators. The enthusiasm of the coordinator for the mathematics curriculum in the public schools (Dr. Sandra Spalt-Fulte) cannot be overstated. Without Spalt-Fulte's vision, dedication, and ability to coordinate diverse groups of stakeholders, the development of CPSM would not have been possible.

It is our hope that these remarks might inspire other educators to take on the task of introducing a course like Creative Problem Solving in Mathematics in their schools. We would be happy to offer our assistance in such an endeavor.

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