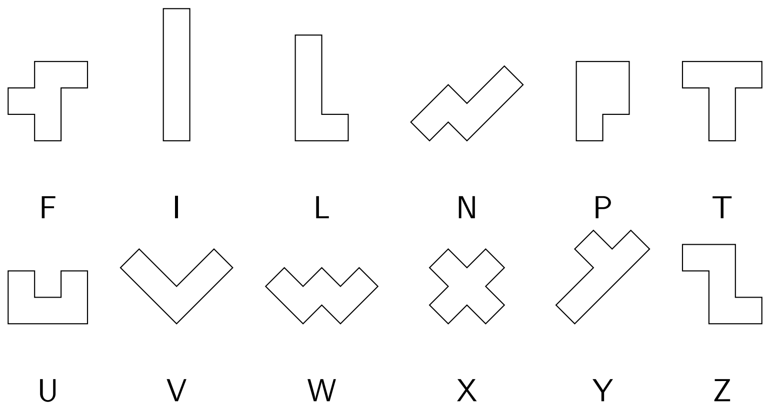
Pentominoes and Problem Solving

Pentominoes are those shapes consisting of five congruent squares joined edge-to-edge. It is not difficult to show that there are only twelve possible pentominoes, shown below. In the literature, each is usually referenced by the letter of the Roman alphabet which it most closely resembles.



What makes pentominoes interesting is that there are few enough of them that they can be easily remembered, but enough of them to be able to build a wide variety of different shapes. They lend themselves to posing and solving a diverse set of problems using problem-solving techniques not encountered in a traditional geometry course. Very importantly, students enjoy playing with pentominoes, so that they are motivated to think about complex problems. And finally, a set can easily be cut out of cardstock (or other heavy paper, such as a manila folder).

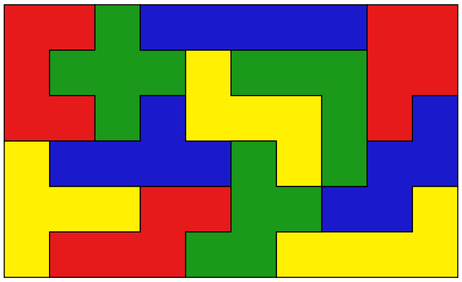
I propose that pentominoes be one theme in the workshops during my visit for several reasons:

1. Problems can be created at any grade level;
2. Most teachers have not worked with pentominoes, so they will learn something new. In addition, they will be able to experience what it is like to be a student again – and this is important when making changes to a curriculum;
3. They can be used in any classroom, since they can be made from paper;
4. Problem-solving with pentominoes specifically addresses Standard M 6.1 regarding reasoning and creative thinking.

What follows are several examples which illustrate the scope of problems which may be created using pentominoes.

Rectangles

The first problem is suitable for Grades 3-5.[1] You are given a set of twelve pentominoes, and you would like to fill up a rectangle using all the pieces. One example is shown below.



What rectangles would be possible to make?

Since there are twelve pentominoes, any possible rectangle must have an area of 60 square units. This allows rectangles of size 1×60, 2×30, 3×20, 4×15, 5×12, 6×10, and the 90-degree rotations of these (such as 60×1).

However, it should be clear that 1×60 and 60×1 rectangles are not possible, since making these rectangles is only possible with 12 I pentominoes.

The 30×2 and 2×30 are also impossible, since many pentominoes (such as X or T) would not be able to fit.

It happens that all other rectangles are possible. Although there is more than one way to make each one (but the 20×3 rectangle has just two solutions), finding any solution is not easy for the inexperienced student of pentominoes. Such a puzzle would be challenging even at the Grade 10–12 level.

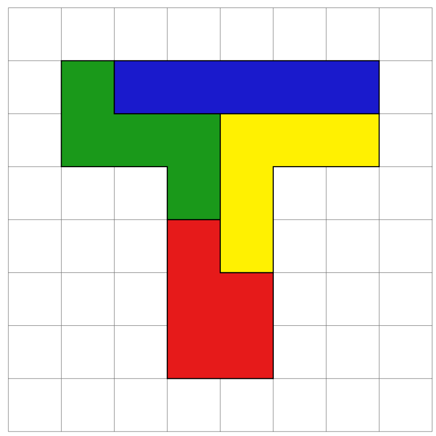
Below are a few problems suitable for Grades 7–9. Once students gain experience with pentominoes, these problems become simpler. But as a first introduction to the subject, they pose a significant challenge.

1. Using the F, I, P, U, and Y pentominoes, make a 5×5 square.
2. Using the I, P, T, V, and W pentominoes, make a 5×5 square.
3. Using the L, T, V, W, and Y pentominoes, make a 5×5 square.
4. Using the N, P, T, V, W, and Y pentominoes, make a 6×5 rectangle.

[1] Without being more familiar with the typical student at a Thai school, it is difficult to assign a particular grade level to any problem. I have given a likely range based on my reading of The Basic Education Core Curriculum.

Replication Problems

It is usually good to start off with simpler problems using a subset of the pentominoes. A good opening problem is that of *doubling* a pentomino; that is, using pentominoes to make one that is doubled along all linear dimensions. As an example, the I, P, V, and Z pentominoes can double the T pentomino:

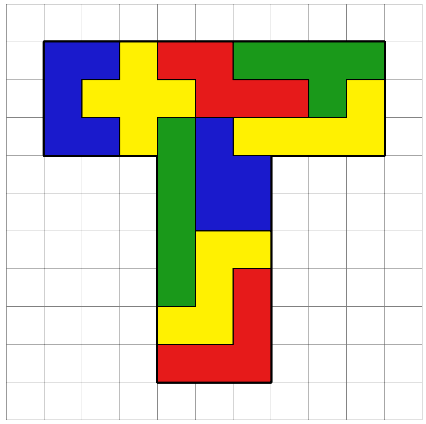


Before seeing such an example, students can be asked the following question: If you wanted to double a pentomino, how many pentominoes would you need? This would be an application of the principle that if linear dimensions of a figure are all scaled by a factor of s, the area is scaled by a factor of s^2. (I looked in the Geometry Standards, but could not find a discussion of areas.)

Such problems are of varying levels of difficulty, depending upon what information is provided.

- Grades 6–7: Double the T pentomino using the I, P, V, and Z pentominoes. In addition, provide a grid with the doubled T pentomino clearly outlined. The grid squares should be the same size as the squares making up the pentominoes.
- Grades 8–9: Double the T pentomino using the I, P, V, and Z pentominoes, but do not provide a grid. This is a conceptually more challenging problem, since students must keep in mind the shape they need to create.
- Grades 10–11: Double the T pentomino. Without specifying a specific subset of pentominoes, the level of challenge increases still further.
- Grades 11–12: What pentominoes *cannot* be doubled? This requires more sophisticated thinking, since students must provide an argument supporting their conjecture.

Of course it is possible to go further and *triple* pentominoes in much the same way that they were doubled. Then nine pentominoes are required, as shown in the following example of tripling the T pentomino.



Here are some example of possible tripling problems.

1. Using all the pentominoes except the U, X, and Y pentominoes, triple the L pentomino.
2. Using all the pentominoes except the T, W, and Z pentominoes, triple the W pentomino.
3. Using all the pentominoes except the L, W, and Y pentominoes, triple the Z pentomino.
4. Using all the pentominoes except the F, I, and X pentominoes, triple the X pentomino.

As it happens, *all* the pentominoes can be tripled. Students can try to show this by finding solutions for each pentomino. The level of difficulty can be changed as with the doubling problems – provide the pieces needed and the grid, provide only the pieces needed, or omit the pieces needed.

Tiling Problems

Problems involving tiling with pentominoes are especially challenging. They may be used in any number of ways – teachers may demonstrate a proof in front of the class as an example of proof technique, students can work in groups or as a class to solve problems, or students can get practice in writing paragraph proofs by giving explanations of results.

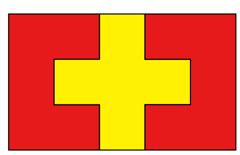
This type of problem addresses standard M 3.2, although it appears that geometry is not included in Grades 10–12. As with the case of Replication Problems, the examples below can be adapted to different grade levels depending upon the hints and suggestions given to students to solve the problems.

Let's begin with a problem accessible to younger students, Grades 3–5. You are given a 15×15 square and a large supply of X pentominoes. How many pentominoes would you need to fill the square? Can you find a solution?

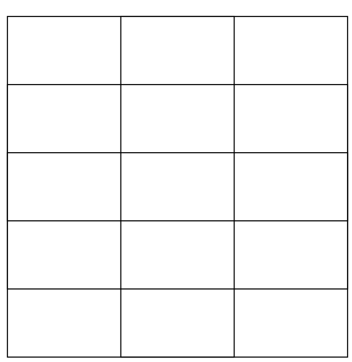
A simple calculation shows that 45 X pentominoes would be needed. However, upon thinking about the problem, it is evident that using only X pentominoes, there is no way to cover a corner square. Thus, there is no solution.

Now let's make the problem a little more involved. Using just X and U pentominoes, is it possible to cover a 15×15 square?

It is possible to solve this problem upon seeing that the X and U pentominoes can be used to make a 5×3 rectangle, as shown below.

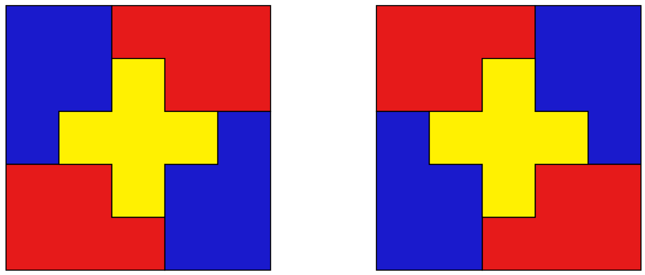


Then it is a matter of using 15 of these units to cover a 15×15 square, as indicated in the following figure.



Taking this problem to a higher level, we ask the following question: Can a 15×15 square be covered using only X and P pentominoes? You must use at least one of each pentomino.

A solution to this problem is a bit more sophisticated, as it involves discovering a figure such as the ones below, which are mirror images of each other:



Thus, once we are able to make a 5×5 square using the X and P pentominoes, we can make a 3×3 array of such squares to complete a 15×15 square.

Taking the problem to the level of Grades 11–12, we ask: Can you tile a 15×15 using only one X pentomino and 44 I pentominoes?

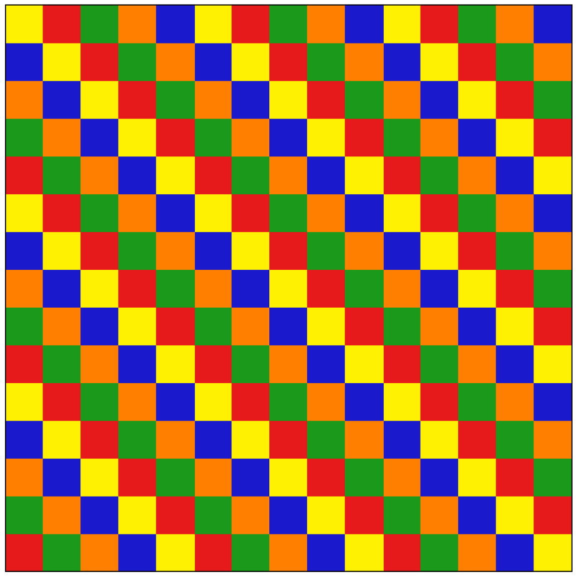
This is a very difficult problem, and involves a proof technique which is standard in working with tiling problems, but which is likely unfamiliar to students. So before giving this problem, it would be necessary to demonstrate a different problem using the same proof technique.

A classic example of this is the following problem: Suppose you have a chessboard, but have removed two opposite corners. You also have 31 dominoes, each of which can be laid to cover two adjacent squares of the chessboard. Is it possible to cover the remaining 62 squares with these 31 dominoes?

Now a chessboard has squares colored alternately black and white. Two opposite corners are always the same color, so assume for the sake of argument that the two corners removed are white. This leaves 32 black squares and 30 white squares to be covered.

However, each domino covers one black and one white squares. Thus, placing 31 dominoes must cover 31 white squares and 31 black squares. Thus, it is not possible to cover the remaining 32 black and 30 white squares with 31 dominoes.

Returning to our problem of the 15×15 square, we may color it using five colors, as shown in the diagram on the next page. Determining how many colors to use in solving any given problem is an important aspect of solving it; the number of colors which will give an easy solution is not always obvious.



Now think about placing the 44 I pentominoes. No matter where one is placed, the I always covers one square of each of the five colors. Thus, no matter where the 44 I pentominoes are placed, they will cover 44 squares of each of the five colors, leaving one square of each of the colors remaining.

Finally, consider placing the X pentomino on the board. It is clear that there is *no* way to place it so that it covers one square of each color – only three different colors are covered. Thus, it is not possible to cover a 15×15 square using just one X pentomino and 44 I pentominoes.

Once problems such as this are solved, other problems can be suggested:

1. Is it possible to cover a 15×15 square using X and I pentominoes, using at least one of each?
2. Is it *ever* possible to cover a rectangle with X and I pentominoes, using at least one of each?
3. If it is possible to cover a rectangle with X and I pentominoes, using at least one of each, what is the rectangle of least area for which this is possible?

These problems are open-ended; the second and third problems may be very difficult indeed. But they help illustrate the point that simpler tiling problems with pentominoes may be given to younger students, while other problems may be so difficult so as to belong to the realm of research-level mathematics.