Problems with Pentominoes

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Objectives

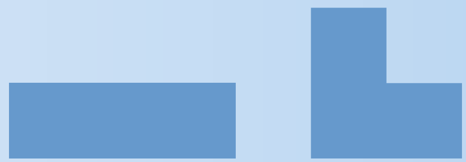
This is a hands-on exploration of problems with pentominoes.

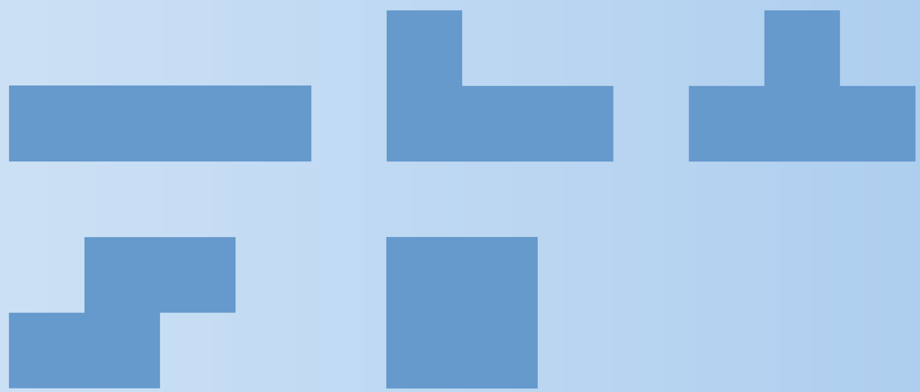
You will gain enough experience to begin designing interesting problems of your own.

Polyominoes

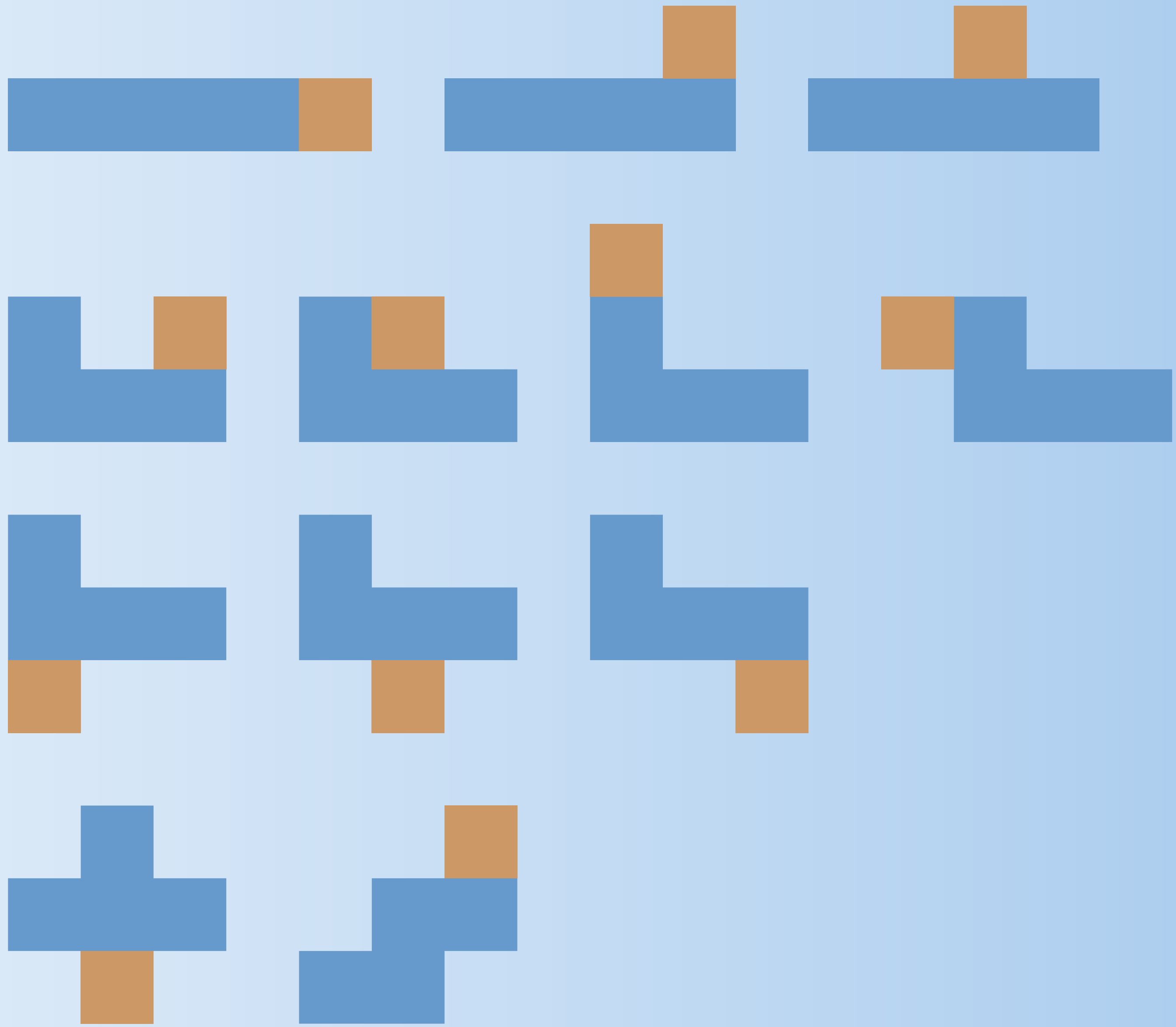
Momomino: 

Domino: 

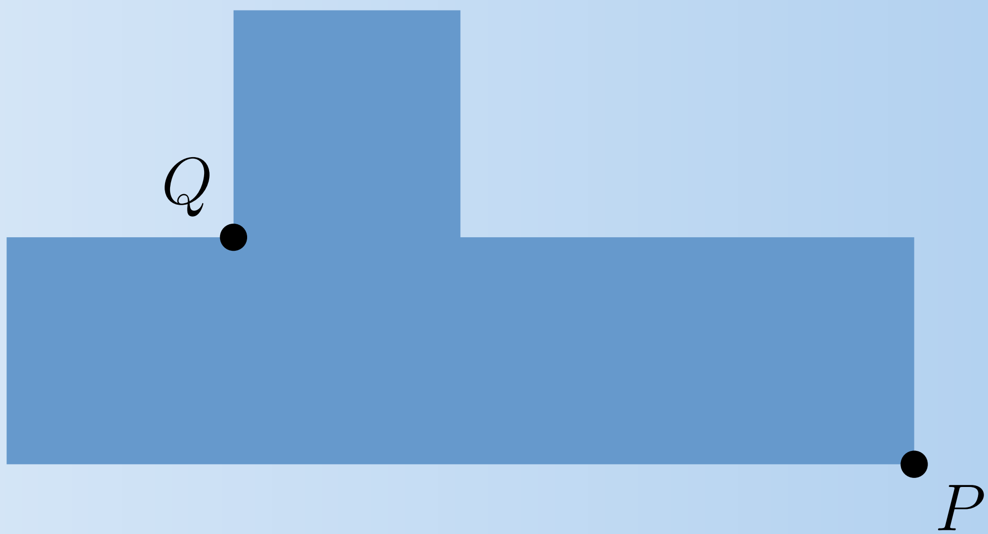
Triominoes: 

Tetrominoes: 

Pentominoes



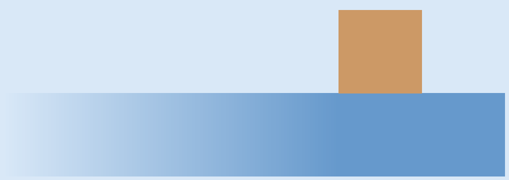
If we call P a *convex* vertex of a pentomino, and Q a *concave* vertex of a pentomino, find a relationship between the numbers of convex vertices and concave vertices on the pentominoes. Then prove your result.



Proof of Invariance

$+0$ convex, $+0$ concave

$+1$ convex, $+1$ concave



$+2$ convex, $+2$ concave

$+0$ convex, $+0$ concave

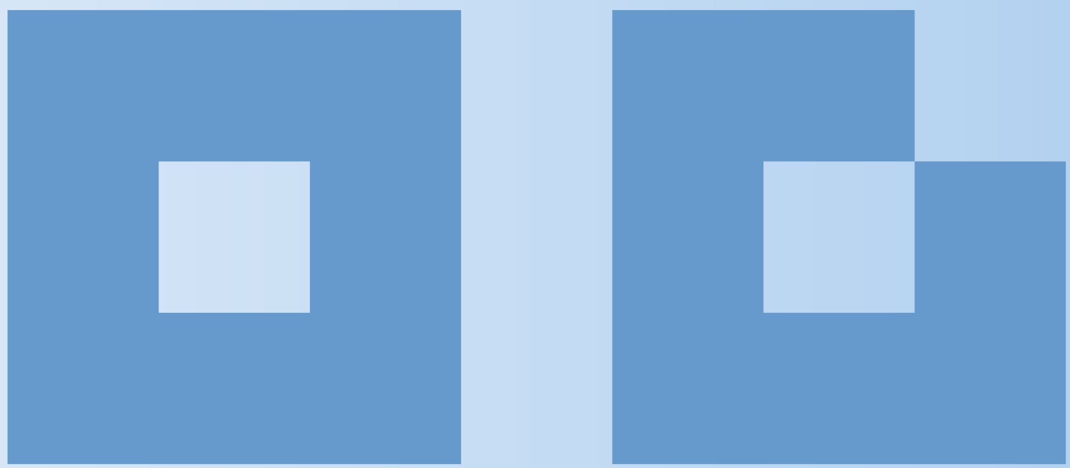
Finishing the Proof

Now imagine creating a pentomino starting from a single square. Note that any pentomino may be built using the moves just described.

But a single square has four convex and zero concave vertices, and so the difference is 4. But adding a square adds the *same* number of convex as concave vertices (if any), and so the difference is still 4.

Generalization is Tricky!

While it is possible to generalize the result, care must be taken. Even if the shape is only made up of unit squares, there may be holes or vertices where squares touch at their corners.

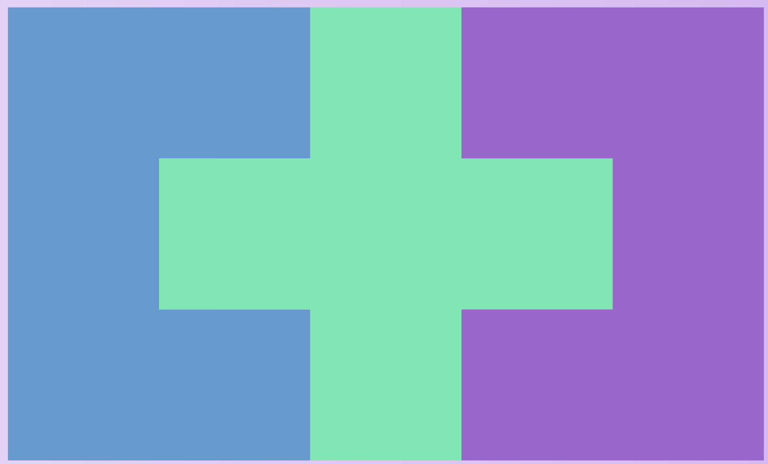


Tiling a Square

Question: Using only the U and X pentominoes, is it possible to tile a 15×15 square?

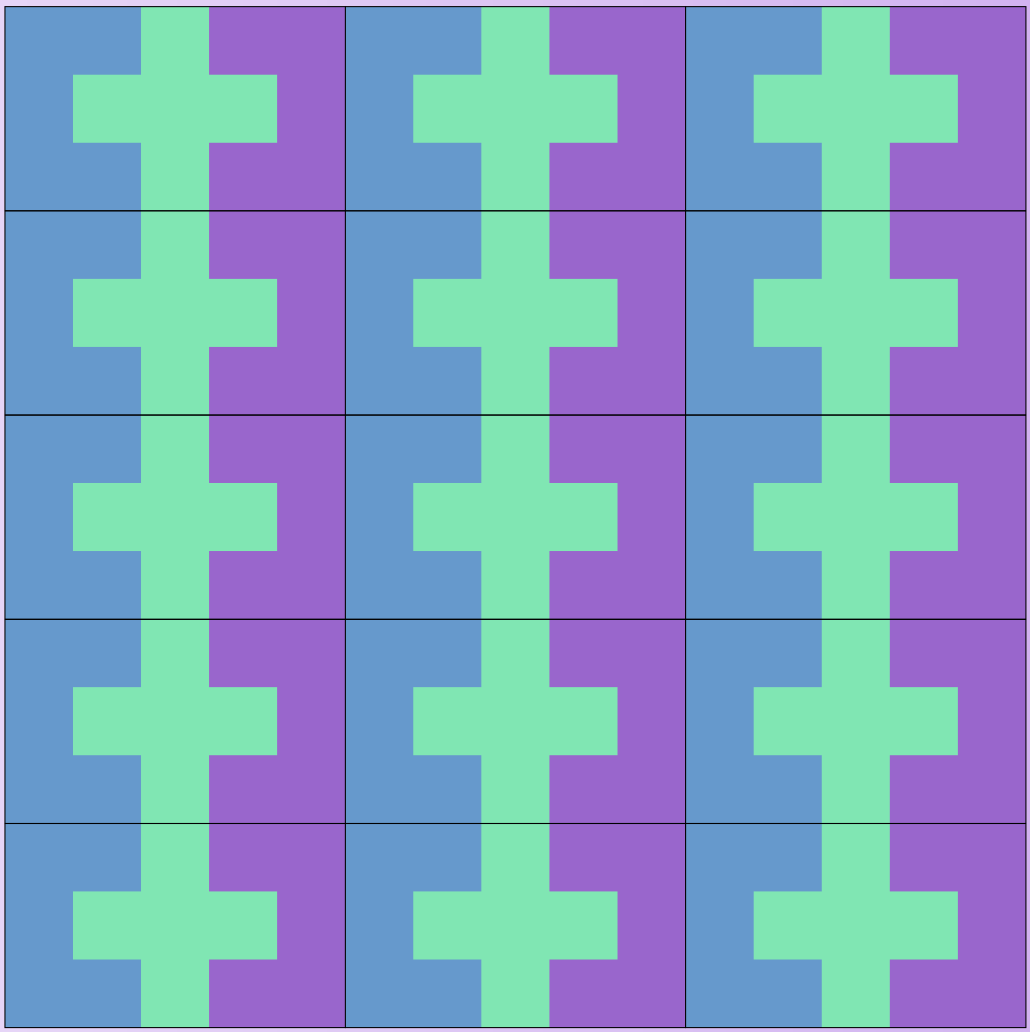
Tiling a Square with U and X

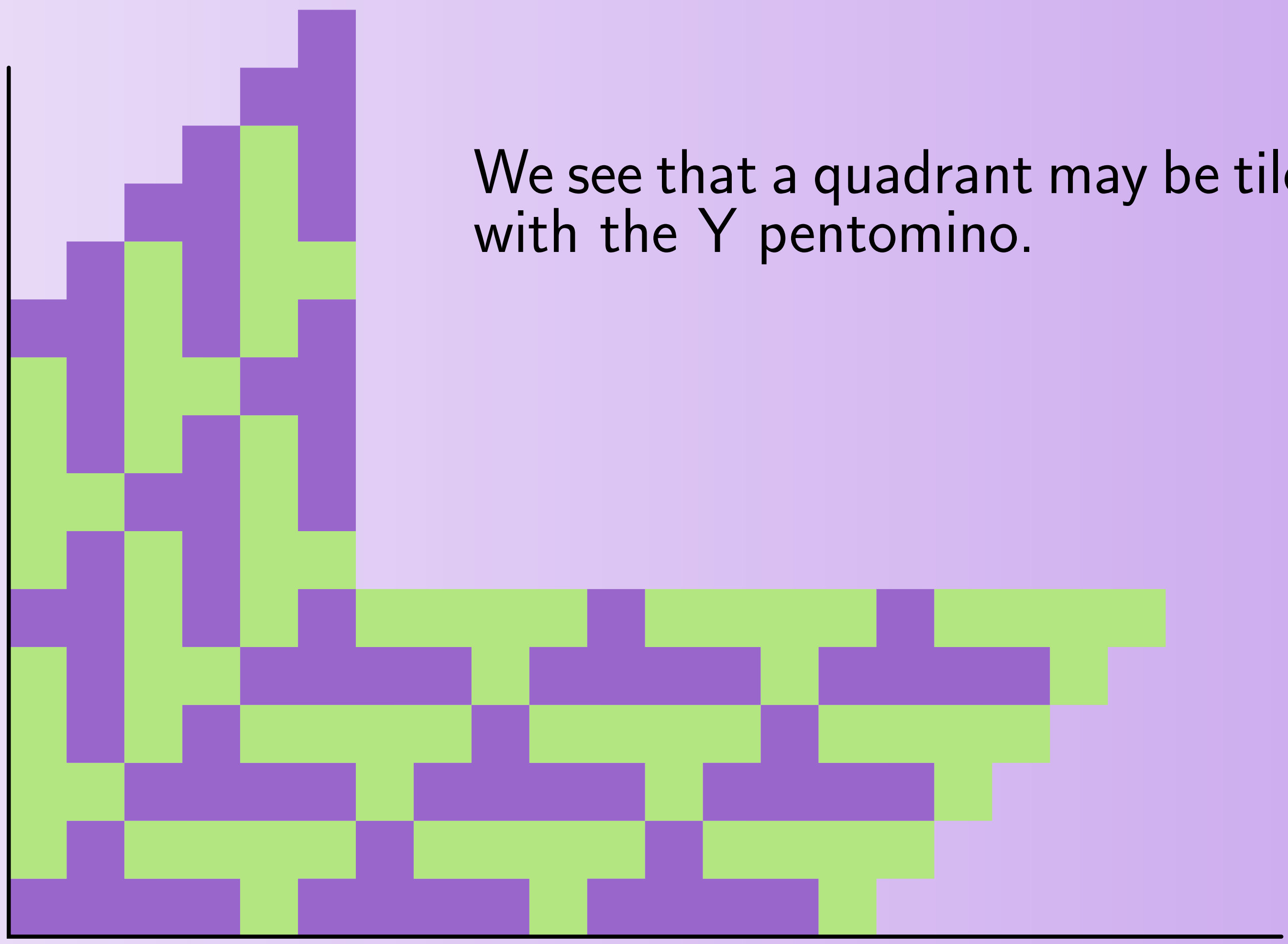
Answer: Yes. One way is to create a smaller unit which can be used to tile a square.



Tiling a Square with U and X

Then complete the tiling of the square.

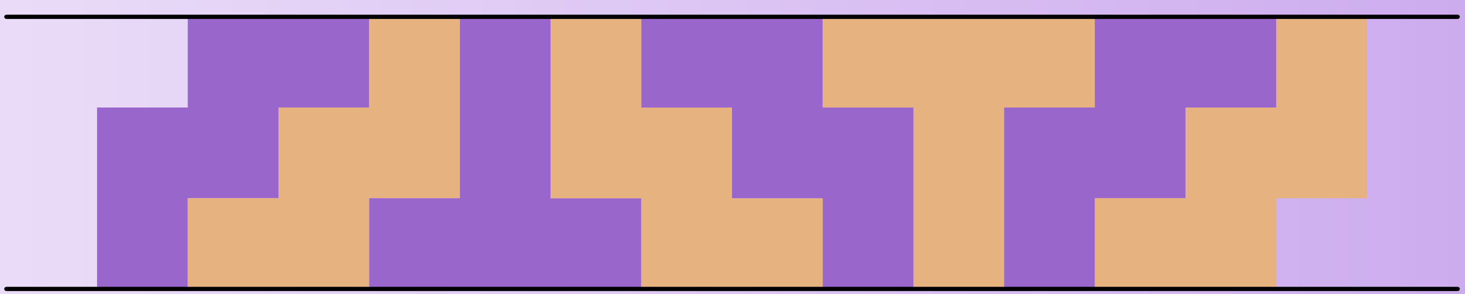




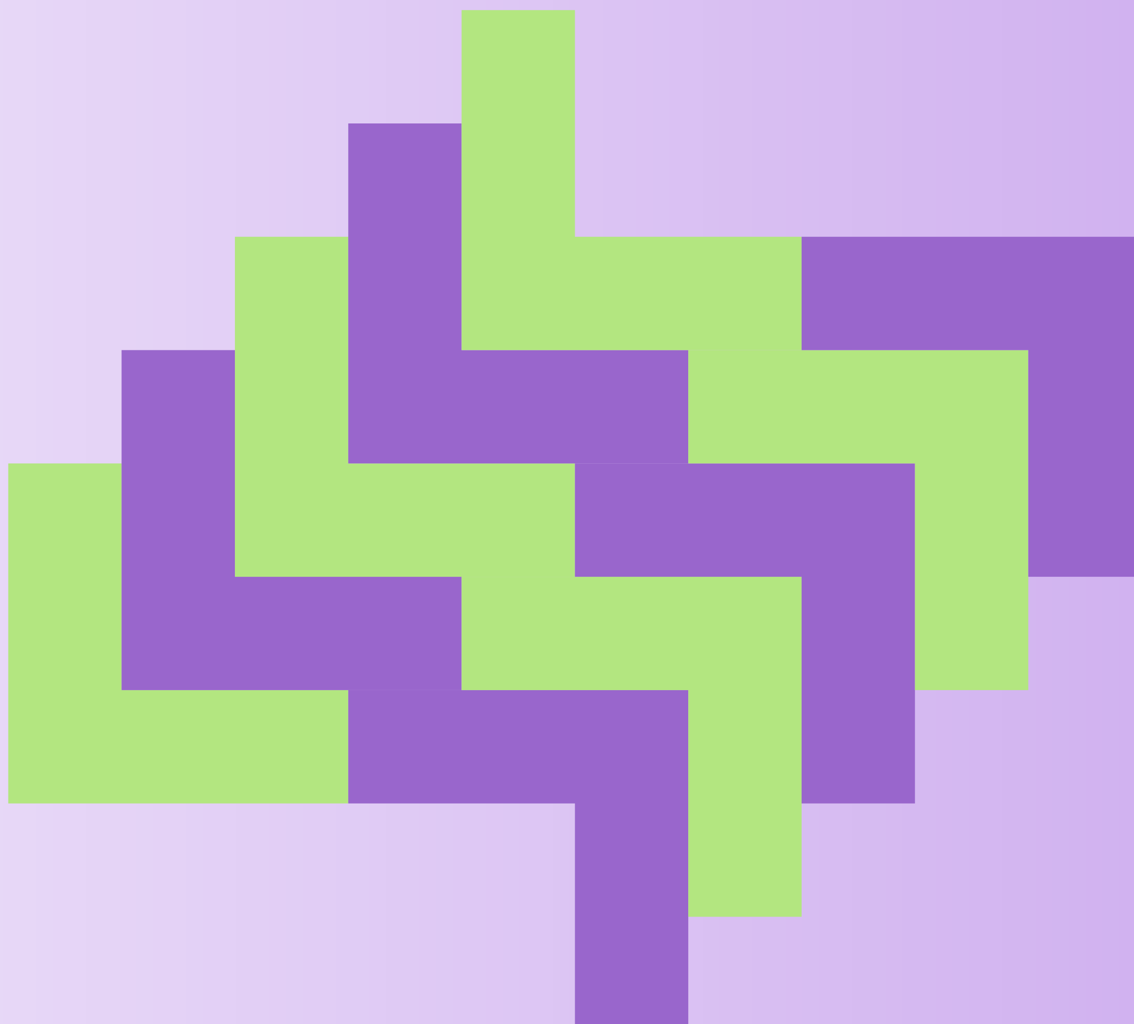
We see that a quadrant may be tiled with the Y pentomino.

Tiling a Strip

Here, we see that the T pentomino and the W pentomino together tile a strip which extends infinitely in both directions. Neither pentomino can tile a strip by itself.



We see that the plane may be tiled with the V pentomino.



An Impossibility Proof

Sometimes, it is *not* possible to create tilings given a set of pentominoes. In that case, an argument must be found which proves the *impossibility* of such a tiling.

Our first example is the following: Show that it is impossible to tile *any* rectangle using only T and Z pentominoes.

Looking at a Corner

One strategy is to observe that some pentominoes *must* go in corners.

Note that excluding rotations and reflections, there is essentially only one way to put a T or Z pentomino in a corner.

In either case, it is impossible to cover the square marked with an $\times$.