Taylor Series

Vincent J. Matsko

Series Like You've Never Seen Them Before

Of course you've probably never worked with Maclaurin or Taylor series before, so in all likelihood, the title is apt.

And the approach is rather different than the usual calculus text. You won't get a series of examples, then a series of exercises which mimic the examples. The exercises are integrated into the text. **DO THE EXERCISES AS YOU ENCOUNTER THEM** (unless instructed otherwise). I mean it. Now some may be easy – and that's good. That means you've retained something from your previous mathematical experience, and you're ready to move on.

But there are some subtleties and surprises. Certain exercises bring up points which are then discussed in the text – so if you skip an important exercise, you'll be a little lost. Or the answer will be explained in the text, and you will not have benefited from the chance to think about the exercise.

If you do get stuck – well, good! Bring questions to class. There are some exercises which are *supposed* to be tricky – and some you won't have the necessary background to solve. Just think of them as opportunities to learn....

The organization is also not what you might expect. The layout is more conversational – we discuss issues as we encounter them in solving particular problems. So the order is definitely different than the usual calculus text.

You'll find lots of things, though, which aren't included in the usual text. All the cool things are purposely *included* – even if they are a little more challenging than usual. And you don't have to wait until the very end to see them. Like *Eulerian Dreamtime,* my particular favorite. It comes as soon as we have the machinery to understand it.

So have fun! And provide feedback as you go along. This is a work in progress, and the idea is to engage you in learning some topics in calculus which are not always presented in an engaging way. So if you've got a suggestion for improvement, let me know. And if you can think of other cool things to add, well, that would be great, too.

As this is the third semester we're using these notes, there are a few acknowledgments. First, the charter BC 2-3 class had many good suggestions which are incorporated into this draft – including more graphics (both in the text and as exercises) and occasional review of antidifferentiation. Thanks also to Matt Tsao who suggested a simpler approach for tackling generating functions.

Contents

1 Recapitulation

It was perhaps not so long ago that we encountered such limits as

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

||| EXERCISE 1: Evaluate this limit using L'Hôpital's rule. How many applications are necessary? Why? □

You might recall that we solved this type of problem earlier using linear (and higher-order) approximations. For example, we knew that to first order,

$$\sin x \approx x,$$

since $y = x$ is the tangent line to $y = \sin x$ at $x = 0$. Antidifferentiating, we see that to second order,

$$-\cos x \approx \frac{1}{2}x^2 + C = \frac{1}{2}x^2 - 1,$$

where C is a constant of integration which we know to be -1 since $-\cos 0 = -1$. Iterating again yields

$$-\sin x \approx \frac{1}{6}x^3 - x + C = \frac{1}{6}x^3 - x,$$

where the constant of integration must be 0. This gives, to third order, the approximation

$$\sin x \approx x - \frac{1}{6}x^3.$$

Hence

$$\sin x - x \approx -\frac{1}{6}x^3,$$

so that in the limit,

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{-x^3/6}{x^3} = -\frac{1}{6}.$$

Of course the fact that this method works out so nicely is that the antidifferentiation is so straightforward. However, consider the limit

$$\lim_{x\to 0} \frac{e^{x^2} - 1}{x^2}.$$

Of course finding the tangent line at $x = 0$ is not too difficult; we see that $y = 1$ does the trick. Thus, near 0,

$$e^{x^2} \approx 1$$

to the first order. (Recall that even though there is no x-term, the tangent line always gives a first-order approximation; in this case, we have $y = 1 + 0x$ since $(e^{x^2})'(0) = 0$.)

But antidifferentiating e^{x^2} is not possible, so continuing on from there doesn't work. We may however, find a first order approximation to $2xe^{x^2}$ and *then* antidifferentiate.

||| Exercise 2: Find the tangent line to $y = 2xe^{x^2}$ at $x = 0$. □

You should have found this to be $y = 2x$. Then, since to first order,

$$2xe^{x^2} \approx 2x,$$

we may antidfferentiate to obtain

$$e^{x^2} \approx x^2 + C = x^2 + 1,$$

where the constant of integration C must clearly be 1. Hence we may easily evaluate the limit

$$\lim_{x\to 0} \frac{e^{x^2} - 1}{x^2} = \lim_{x\to 0} \frac{x^2 + 1 - 1}{x^2} = 1.$$

It might seem a bit of a stretch to claim that the limits just written must be the same. But *assuming* that we can approximate e^{x^2} by polynomials of higher and higher degree (and we will be looking at this process in great detail soon), it is really not so farfetched. For example, suppose we found a third-degree approximation:

$$e^{x^2} \approx kx^3 + x^2 + 1.$$

Would this produce a different result? No, since

$$\lim_{x\to 0} \frac{kx^3 + x^2 + 1 - 1}{x^2}$$

is still 1. Adding higher terms will not, in fact, change evaluating this type of limit.

Still, it is not so simple to use the same technique to evaluate

$$\lim_{x\to 0} \frac{e^{x^2} - x^2 - 1}{x^4}.$$

||| Exercise 3: Evaluate this limit using L'Hôpital's rule. □

How might this be done using approximations? Since we require a fourth-order approximation, this would require finding a linear approximation to the third derivative of e^{x^2}, and then antidifferentiating three times.

||| Exercise 4: Carry out the procedure just described. □

Whew! You might be thinking that there *must* be an easier way – and there is. Much of what will occupy us in the near future concerns itself with an easier and more general approach. Not only will we be able to evaluate complex limits more easily, but we will see many other applications of these methods – including, finally (!), the answer to questions such as, "Just how good is the trapezoidal approximation, anyway?"

2 A New Approach

Let's reexamine the first limit we attempted to evaluate:

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}.$$

We were seeking a third-order approximation, so we began with a first-order approximation and antidifferentiated twice.

But we *know* what a third-order approximation looks like near 0:

$$\sin x \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

That is, we have a cubic polynomial – a polynomial of degree 3 – some of whose coefficients may be 0. Moreover, we know that $a_0 = 0$ since we must have $\sin 0 = 0$.

Now how can we find the other coefficients? Recall that when we began with a lower-order approximation, we antidifferentiated and then found the constant of integration. Since we are beginning with the cubic approximation, let's try differentiating instead, so that

$$\cos x \approx a_1 + 2a_2 x + 3a_3 x^2.$$

Evaluating at 0, we see that $a_1 = 1$. Continuing the process, we have

$$-\sin x \approx 2a_2 + 6a_3 x,$$

so that $a_2 = 0$. Finally, we have

$$-\cos x \approx 6a_3,$$

so that $a_3 = -1/6$. Thus, we have

$$\sin x \approx x - \frac{1}{6}x^3$$

as our third-order approximation, as expected. You should take a few moments and compare this method with the first one, and convince yourself that they are in some sense equivalent – it really just depends on where you start, either at the lower-order end or the higher-order end.

Let us take a moment to consider the significance of the "0" in "$\lim\limits_{x\to 0}$." Given the range of the sine function, the approximation $\sin x \approx x - \frac{1}{6}x^3$ is certainly not very good, say, at $x = 10$. It should be clear that beyond a certain point, in fact, the approximation is virtually useless. So although we have found a third-order approximation to $\sin x$, it is equally important to ask when the approximation is a good one – given we know what "good" means.

We will discuss such error analysis in detail later. For now:

||| Exercise 5: Graph the functions $y = \sin x$ and $y = x - \frac{1}{6}x^3$ with your favorite software. It should be clear that close to 0, the graphs are nearly identical. Visually inspect the graphs, and give an interval for which the polynomial is a good approximation to the trigonometric function. □

Now it might strike you that there is nothing about "$\sin x$" which makes the above method work. In fact, you should be able so find other approximations in a similar way.

||| Exercise 6: By using this method, find a fourth-order approximation to $y = e^{x^2}$. By visually inspecting the graphs, give an interval on which this is a reasonable approximation.

□

In the future, you will sometimes be asked to graph various approximations and comment on them. However, you should always feel free to graph functions even if not explicitly asked if you think it would help your understanding of a particular concept. Use technology wisely!

||| Exercise 7: Using this method, find a fourth-order approximation to $y = f(x)$ near $x = 0$. □

There is clearly something interesting going on here – and a definite pattern to the coefficients. Essentially all we need to do is evaluate successive derivatives at 0. This is nice because, as we have seen, derivatives are generally much easier to evaluate than antiderivatives. And we can – evidently – get any order approximation we wish by taking more and more derivatives.

By now, several questions may have arisen in your mind:

1. Why would we want to make such approximations? Is it just to evaluate limits?
2. What is a "good" approximation? How close to 0 do we need to be in order to get a "good" approximation?
3. Isn't it easier just to use a calculator or *Mathematica*?
4. Can we estimate the order of these approximations (as we discussed with Euler)?

3 Working with Series

In looking at the previous result, there does not seem to be anything which prevents us from making better and better approximations to a function, say $\sin x$, as necessary.

||| Exercise 8: Find a ninth-order approximation to $\sin x$. Then find a tenth-order approximation. □

You should have had little difficulty finding that a ninth-order approximation to $\sin x$ is given by

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9.$$

It turns out that this is *also* a tenth-order approximation, since the coefficient of x^{10} turns out to be 0 in this case. It is important to note that although there are only five monomials in this expansion, we still call it a ninth- (or tenth-) order approximation. The order of the approximation is determined by the highest-degree monomial present.

But how would you know whether the approximation given is ninth-order or tenth-order just by looking at it? Although you might consider it as either, more information is usually given in order to make the context clear. If you did not know the approximation was that for $\sin x$, you would have to call this a ninth-order approximation.

Of course one may make higher- and higher-order approximations – and there is a convenient way to denote this using series notation. To indicate the pattern for arbitrary order, we say that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

Now take a moment to study this notation carefully. Convince yourself that the "$(-1)^k$" term does indeed alternate signs as needed, and that the "$2k+1$" terms do indeed give the odd powers.

Using this notation, we may write the ninth-order approximation to $\sin x$ as

$$\sin x \approx \sum_{k=0}^{4} \frac{(-1)^k}{(2k+1)!}x^{2k+1}.$$

Please note: although this is a *ninth*-order approximation, the variable k ranges only from 0 to 4. You must always remember that the order of the approximation is the degree of the highest monomial present.

Before we discuss series in more detail, pause for a moment to think about this example. It should be clear that the infinite series approximation exactly computes $\sin 0$. For small values of x, say $\pi/10$, it also appears that the higher powers of a smaller number, together with the factorials on the denominator, indicate that the series would approach $\sin(\pi/10)$ as you go further out in the series. But what about $\sin \pi$? $\sin 100$? $\sin 10^6$? Of course sin is periodic – and with the familiar properties of the sine function, it is clear that if we know the values of sine on $[0, \pi/2]$, we can determine $\sin x$ for any x. The important question is: *could* we determine $\sin 10^6$ using the series if we so desired? And how do we answer this question?

We need to do a little more work to answer this sort of question in general. Before we do so, though, it is important to review our knowledge about series and series notation. It has probably been a little while since you've worked with them. We'll get to some exercises, but let's review a few definitions.

Definition 1: Let a sequence a_k be given, $k \geq 1$. The sequence

$$S_n = \sum_{k=1}^{n} a_k$$

is called the **sequence of partial sums** of the sequence a_k, or the **series** corresponding to a_k. The series

$$\sum_{k=1}^{\infty} a_k$$

is said to **converge** if $\lim_{n\to\infty} S_n$ exists, and this limit is called the **sum** of the series.

Now on to the exercises.

||| Exercise 9: Write out the terms of

$$\sum_{j=3}^{7} (-1)^j (5j - 10).$$

□

||| Exercise 10: Write as a finite series (that is, using Σ–notation):

$$-3 + 1 + 5 + 9 + \cdots + 21.$$

□

||| Exercise 11: Prove that the sum of a finite arithmetic series is given by

$$\frac{n(a + l)}{2},$$

where n represents the number of terms, a denotes the first term of the series, and l denotes the last term. □

||| Exercise 12: Can you use the formula you just derived to sum the first two finite series? Why or why not? □

||| Exercise 13: What is the formula for summing an infinite arithmetic series? □

||| Exercise 14: Write out the series

$$\sum_{k=2}^{5} \frac{(-1)^k}{3^k}.$$

□

||| Exercise 15: Write as a finite series:

$$-\frac{1}{2} + 1 - 2 + 4 - 8 + 16 - 32.$$

□

||| Exercise 16: Write as a finite series:

$$-\frac{1}{2} + 1 - 2 + 4 - 8 + 16 - 32 + \cdots + (-1)^n 2^n.$$

□

||| Exercise 17: Consider a finite series given by

$$S = \sum_{k=0}^{n-1} ar^k.$$

(This is standard notation for summing a finite geometric series.) Evaluate using the usual technique of finding the difference $S - rS$ and performing the appropriate algebraic steps. What restrictions are necessary on r for this formula to be valid? What is the sum of the finite series when this formula does not apply? □

||| Exercise 18: Sum the finite geometric series you've already encountered using this formula. □

||| Exercise 19: Consider the infinite sum

$$\sum_{k=0}^{\infty} \frac{1}{2^k}.$$

This is an *infinite series.* Recall that there is a very particular way of evaluating such a series.

1. First, write the sequence of partial sums
$$S_n = \sum_{k=0}^{n} \frac{1}{2^k}.$$
Evaluate S_n.
2. Does the sequence S_n converge? How do you know? To what does S_n converge?
3. Discuss the convergence of
$$\sum_{k=0}^{\infty} \frac{1}{2^k + 1}.$$

□

||| Exercise 20: What is the formula for summing the infinite geometric series
$$S = \sum_{k=0}^{\infty} ar^k?$$
What restrictions are necessary on a and r? □

||| Exercise 21: Consider the following series:
$$\sum_{k=0}^{n} (-1)^k \left(\frac{2}{5}\right)^k.$$
Give a formula for S_n, and discuss the convergence of this series. □

||| Exercise 22: Discuss the convergence of
$$\sum_{j=0}^{\infty} \frac{3^j - 2^j}{4^j}.$$

□

||| Exercise 23: Consider the sum
$$\sum_{k=0}^{\infty} e^{kx}.$$

1. For what values of x does this series converge?
2. For these values of x, find the sum of the series.

□

||| Exercise 24: Discuss the convergence of

$$\sum_{n=0}^{\infty}\left(\frac{i}{2}\right)^n.$$

□

||| Exercise 25: For the following series,

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right),$$

find an expression for the sequence of partial sums S_n. Prove your result is valid using mathematical induction. Then determine if the series converges. □

||| Exercise 26: Evaluate:

$$\sum_{k=1}^{\infty}\frac{1}{k(k+2)}.$$

□

||| Exercise 27: Find

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}.$$

□

||| Exercise 28: Discuss the convergence of

$$\sum_{k=0}^{\infty}\frac{1}{\sqrt{k}+\sqrt{k+1}}.$$

□

This series is a good example of why you must be careful when considering telescoping series. It is tempting to think that "everything cancels," and thus any telescoping series must converge. But it is crucial to remember that the convergence of a series is determined by looking at the sequence of partial sums.

This does not mean that we can't use our intuition when it comes to telescoping series. The idea is this: essentially, telescoping series converge when the terms which do not cancel go to 0. And while this is a reasonable intuition, to be thorough you must investigate the corresponding sequence of partial sums.

||| EXERCISE 29: Find all x which satisfy

$$\sum_{k=0}^{\infty} 3x^{2k} = \sum_{k=0}^{\infty} 2x^{3k}.$$

□

||| EXERCISE 30: Determine the number of real solutions to the following equation in x:

$$\sum_{k=0}^{\infty} x^{2k+3} = \sum_{k=0}^{\infty} x^{3k+2}.$$

□

||| EXERCISE 31: Determine if the following series converges:

$$\sum_{n=1}^{\infty} \frac{e^{\sin n}}{e^{\cos n}}.$$

□

||| EXERCISE 32: Given that

$$\sum_{n=1}^{\infty} = \frac{\pi^2}{6},$$

find the sum of the following series:

$$\sum_{n=3}^{\infty} \frac{1}{n^2(n-2)}.$$

□

||| EXERCISE 33: A sequence a is said to *converge quadratically* if $\lim\limits_{n\to\infty} \dfrac{a_n}{n^2}$ exists. For the following, prove or provide a counterexample. Assume that the sequences a and b satisfy $a_n > 0$ and $b_n > 0$ for all $n > 0$. Notationally, the terms of the sequence ab are $a_n b_n$.

1. If a converges quadratically and b converges, then ab converges quadratically.
2. If ab converges quadratically and a converges quadratically, then b converges.
3. If ab converges quadratically and b converges, then a converges quadratically.

$\square$

||| EXERCISE 34: Recall that a sequence is simply a function whose domain is $\mathbb{N}$. (Here, $0 \in \mathbb{N}$.) We say that $a : \mathbb{Z} \to \mathbb{R}$ is a *doubly-indexed sequence.* The *doubly-indexed series*

$$\sum_{n=-\infty}^{\infty} a_n$$

is said to converge if the sequence of partial sums, given by

$$S_n = \sum_{k=-n}^{n} a_k,$$

converges. Given a double-indexed sequence a, we define the sequences $a^+ : \mathbb{N} \to \mathbb{R}$ and $a^- : \mathbb{N} \to \mathbb{R}$ by

$$a_n^+ = a_n, \quad a_n^- = a_{-n}, \quad n \in \mathbb{N}.$$

For each of the two statements below, prove it or provide a counterexample. Here, a is a doubly-indexed sequence.

1. If the series $\displaystyle\sum_{n=-\infty}^{\infty} a_n$ converges, then so do both the series $\displaystyle\sum_{n=0}^{\infty} a_n^+$ and $\displaystyle\sum_{n=0}^{\infty} a_n^-$.
2. If both the series $\displaystyle\sum_{n=0}^{\infty} a_n^+$ and $\displaystyle\sum_{n=0}^{\infty} a_n^-$ converge, then so does the series $\displaystyle\sum_{n=-\infty}^{\infty} a_n$.

$\square$

||| EXERCISE 35: Discuss the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

$\square$

4 The Direct Comparison Test

In looking at

$$\sum_{k=0}^{\infty} \frac{1}{2^k+1}$$

in the previous section, it may have seemed "obvious" that this series converged. Our intuition is sound: we know that for $k \geq 0$,

$$\frac{1}{2^k+1} < \frac{1}{2^k},$$

and since

$$\sum_{k=0}^{\infty} \frac{1}{2^k}$$

converges (being a convergent geometric series), and since the given series can't, in some sense, get any larger than this, it must converge as well.

As it happens, this is true – but in some ways, not really all that obvious. Let's take a moment to see why. Consider the partial sums

$$S_n = \sum_{k=0}^{n} \frac{1}{2^k+1}.$$

We clearly have

$$S_n < \sum_{k=0}^{n} \frac{1}{2^k} < \sum_{k=0}^{\infty} \frac{1}{2^k} = 2,$$

and it is also clear that $S_n < S_{n+1}$; that is, S is an increasing sequence. Because $S_n < 2$ for $n \geq 0$, we say that the sequence S is *bounded above by* 2.

Now the Monotone Convergence Theorem in Real Analysis states that any increasing sequence which is bounded above *must* converge. But why do we need a theorem for this? Isn't it obvious?

Let's restate the Monotone Convergence Theorem: any increasing sequence of *real* numbers, bounded above, must converge to a *real* number. The emphasis here is on the fact that the numbers are *real*.

This distinction is rather critical. Is it true, for example, that any sequence of *rational* numbers which is bounded above must converge to a *rational* number? This is clearly false – just think of the decimal approximations to π, for example:

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \ldots.$$

This is an increasing sequence of rational numbers which is bounded above and which converges to π, which is *not* rational.

This example lies at the heart of Real Analysis – the difference between rational numbers and real numbers, and the theorems which are valid for one set of numbers, but not the other. It's really interesting stuff. Everyone should take a course in Real Analysis....

What this means for us, though, is that we may use our intuition:

> THEOREM 2: (**Direct Comparison Test**) Suppose a_n and b_n, $n \geq 1$, are sequences satisfying
> $$0 \leq a_n \leq b_n.$$
> Then if $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges. However, if $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ also diverges.

Now there are a few important things to note here. First, the sequences begin with $n = 1$. Of course they may begin at $n = 0$ or any other convenient place, but we need to give *some* starting place for specificity. Second, who said anything about divergence? Well, think about it. If you know that a series diverges, and every term of a second series is larger that the first, then that series must diverge as well. Thus, the Direct Comparison Test gives us a way to determine divergence as well. At least is some cases....

||| EXERCISE 36: Suppose you know that for all $n \geq 1$, we have $a_n \leq n$ and $b_n \geq 2^{-n}$. What can you conclude about $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$? □

||| EXERCISE 37: Show that
$$\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$$
converges. Find a fraction which gives the sum of this series to within two decimal places. (BONUS: Do this *without* using a calculator!) □

You likely didn't find showing the convergence all that troublesome, but finding an approximation was likely a little tougher. This is typically the issue: most series "close" to easily calculable series are notoriously difficult to evaluate. Of course it is obvious (after invoking a little *Mathematica*) that the sum of this series is in fact
$$\frac{-\log\left(\frac{3}{2}\right) + \psi_{1/3}^{(0)}\left(-\frac{i\pi}{\log(3)}\right)}{\log(3)}.$$

||| EXERCISE 38: Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}.$$

1. Indicate how you can determine the convergence of this series knowing that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2. Find the sum of this series.

□

Were you able to use the Direct Comparison Test for the previous exercise? It *is* possible, but you might need to do a little algebra. The problem, clearly, is that

$$\frac{1}{n^2 - 1} > \frac{1}{n^2},$$

so that the inequality is in the wrong direction. But by observing that $n^2 < 2(n^2 - 1)$ for $n \geq 2$, we may write the inequality

$$\frac{1}{n^2 - 1} < \frac{2}{n^2}$$

for $n \geq 2$, with the inequality now going in the correct direction for using the Direct Comparison Test. This is legitimate, since if a series converges, so must twice the series – or for that matter, any other multiple of the series. So sometimes the trick is to find the appropriate multiple – making sure to algebraically *prove* it gives the necessary inequality.

||| EXERCISE 39: Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$$

converges using the Direct Comparison Test and an appropriate multiple of a known convergent series. □

A nice application of the Direct Comparison Test allows us to determine whether the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n}$$

converges or diverges. The trick here (as is usually the case with the Direct Comparison Test) is to pick the right series to compare to.

$$\begin{aligned} 1 &\geq \frac{1}{2}, \\ \frac{1}{2} &\geq \frac{1}{2}, \\ \frac{1}{3}+\frac{1}{4} &\geq \frac{1}{4}+\frac{1}{4}=\frac{1}{2}, \\ \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} &\geq \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}, \\ \frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{15}+\frac{1}{16} &\geq \frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}+\frac{1}{16}=\frac{1}{2}. \end{aligned}$$

Thus, it is clear that

$$\sum_{n=1}^{16} \frac{1}{n} \geq \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{5}{2}.$$

Of course the next 16 terms are each greater than or equal to $1/32$, so that

$$\sum_{n=1}^{32} \frac{1}{n} \geq \frac{5}{2}+16 \cdot \frac{1}{32}=3.$$

The purist will wish to prove by mathematical induction that

$$\sum_{n=1}^{2^M} \frac{1}{n} \geq \frac{M+1}{2}.$$

In any case, it is clear that the partial sums of the harmonic series increase without bound, and thus the series diverges.

||| Exercise 40: Given that we now know that the harmonic series diverges, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p},$$

where $p>0$. Decide to the extent possible, using the Direct Comparison Test, for which p this series converges and/or diverges. □

||| Exercise 41: Using the method described in the text for showing that the harmonic series diverges; that is, by choosing an appropriate series for comparison, show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges. □

||| EXERCISE 42: Let strictly positive series a and b be given. Suppose that $\sum_{n=1}^{\infty} a_n$ converges, and that

$$|b_n - a_n| < \frac{1}{2^n}$$

for all $n \geq 1$. Must $\sum_{n=1}^{\infty} b_n$ converge? □

5 Maclaurin Series

OK, now we've refreshed our memory about series and discuss some basic issues regarding their convergence. A few important comments are in order:

1. A series converges only if the sequence of partial sums converges. This implies a definite *order* in which the terms are added. Let's consider an example which we know converges:
$$\sum_{k=0}^{\infty}\frac{1}{2^k}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^k}+\cdots$$
Now I can rewrite each term to create the following series:
$$1+1-\frac{1}{2}+1-\frac{3}{4}+1-\frac{7}{8}+1-\frac{15}{16}+1-\frac{31}{32}+\cdots+1-\frac{2^k-1}{2^k}+\cdots$$
Now one may naively think, "OK, I'll just add up all the terms with a "+" in front of them, and then subtract all the terms with a "−" in front." But this would first result in adding an infinite number of ones! This of course diverges, as does the sum of the negative terms.

   Of course if the number of terms is finite, the order in which they are added is irrelevant – you'll always get the same sum. But as we have just seen, you can't just selectively extract certain terms from an *infinite* series and add them. The definition of the sum of an infinite series as a sequence of partial sums is important for just this reason. In other words, *order matters* in summing a series. (But only sometimes...more on that later....)

2. Deciding whether a series converges – and if it does, to what it converges – are two *very* different questions. We will spend quite a bit of effort in learning ways to decide if a series converges – but in very few cases will we be able to actually sum the series. In most cases, the sum needs to be done numerically.

   You will notice that our brief review included summing finite arithmetic series, as well as finite and infinite geometric series. Typically, these are the *only* series which are easy to evaluate. If you doubt this, try summing
$$\sum_{k=1}^{\infty}\frac{1}{k2^k},$$
or perhaps
$$\sum_{k=0}^{\infty}\frac{k}{2^k}.$$
It's not so easy...but it *is* possible. (In fact, we *will* see how to sum these series, but that will need to wait a little bit. There's too much else to do at the moment.)

So recall our previous work, where we claimed that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

Please notice – carefully – the use of the "=" in this equation. This is no longer an approximation, but we still haven't determined for what x we actually have equality. This way of writing $\sin x$ – as an infinite sum of monomials – is called a **Maclaurin** series. In order to have some concrete examples to work with, you should find the Maclaurin series for the following functions. Write them using the same notation we used for writing $\sin x$.

||| Exercise 43: Write as Maclaurin series:

1. $\sinh x$
2. $\cos x$
3. $\cosh x$
4. e^x
5. $\dfrac{1}{1-x}$
6. $\dfrac{1}{1+x}$

□

||| Exercise 44: Recall that given a function $f(x)$, we may define the **even** and **odd** parts of f, $E(x)$ and $O(x)$, by

$$E(x) = \frac{f(x) + f(-x)}{2}, \quad O(x) = \frac{f(x) - f(-x)}{2}.$$

In light of these definitions, comment on the Maclaurin series you just derived. □

Many of the Maclaurin series you encountered in the previous few exercises recur so frequently, it is worthwhile to make sure your list is complete and accurate:

Important Maclaurin Series 3:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+\cdots+f^{(k)}(0)\frac{x^k}{k!}+\cdots \tag{1}$$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^k}{k!}+\cdots \tag{2}$$

$$\sinh x=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots+\frac{x^{2k+1}}{(2k+1)!}+\cdots \tag{3}$$

$$\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)!}+\cdots \tag{4}$$

$$\cosh x=\sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots+\frac{x^{2k}}{(2k)!}+\cdots \tag{5}$$

$$\cos x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^k x^{2k}}{(2n)!}+\cdots \tag{6}$$

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+\cdots+x^k+\cdots \tag{7}$$

$$\frac{1}{1+x}=\sum_{k=0}^{\infty}(-1)^k x^k=1-x+x^2-x^3+\cdots+(-1)^k x^k+\cdots \tag{8}$$

||| Exercise 45: Find the Maclaurin series for $f(x)=\sin(5x)$ using equation (1) above. Then find the Maclaurin series by direct substitution in (4). What do you notice? □

Now it's time to foreshadow a bit of what's coming next. We'll look at a few examples, and perhaps raise more questions than we answer. But we *do* know enough calculus at this point to ask some interesting questions....

As our first example, consider the Maclaurin series for $g(x)=\dfrac{1}{1+x^2}$. You might be tempted to dive in by taking some derivatives, but by taking a moment and looking back at the Important Maclaurin Series, you might recognize that substituting x^2 in for x in formula (8) will do nicely, giving

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots$$

Now what happens when we antidifferentiate both sides of this equation? We get

$$\arctan(x)+C=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

The C is *always* important in undertaking this type of calculation (as we saw at the very beginning of our discussion of series). We check that $C = 0$ by considering $x = 0$.

Now what happens when $x = 1$? It appears that

$$\arctan(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Seems intriguing that it is so easy to calculate π, doesn't it?

Well, yes, but we *did* skip a few important details. For example, if we know that a function is equal to an *infinite* sum of monomials, is an antiderivative of the function equal to an *infinite* sum of antiderivatives of these monomials? In the finite case, the answer is certainly "yes." But it is not so obvious in the infinite case.

And how do we know we can just put in $x = 1$? A moment's thought reveals that trying $x = 2$, for example, is highly problematic – the terms we alternately add and subtract grow exponentially. Just what values of x *can* we substitute in our series?

Now as it happens, it is in fact legitimate to perform this calculation. But this raises another point: is this a good way to evaluate π? How many terms will it take to get good accuracy? Using *Mathematica,* we see that using the first 1,000 terms of this series gives $\pi \approx 3.14259$. So 1,000 terms gives us only two decimal places of accuracy? This does *not* seem to be a very efficient way to calculate π.

Now for our second example: we will sum the series

$$\sum_{k=1}^{\infty} \frac{k+1}{2^k}.$$

Although we are fairly good at summing geometric series, there is that k in the numerator – and even if we break the sum into two, it still remains. How do we handle such series?

We begin by thinking in terms of geometric series. For example, using $x = 1/2$ in formula (7) above will give us a 2^k term in the denominator. But how to obtain a k in the numerator? We know that differentiating in some sense "brings down" the exponents in polynomials, so why not try it here? Of course skipping the important detail of whether it is a valid step to differentiate an infinite sum that way....

So let's write

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots .$$

Now differentiate, obtaining

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

How close does this get us? Using $x = 1/2$ gives

$$4 = 1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \cdots.$$

Thus,

$$\sum_{k=1}^{\infty} \frac{k+1}{2^k} = \frac{2}{2^1} + \frac{3}{2^2} + \frac{4}{2^3} + \frac{5}{2^4} + \cdots = 3.$$

Neat! This is only one such series which can be summed using this technique. More details about the legitimacy of performing the various steps will be gradually addressed. But the point is that many problems previously out of reach are now accessible using Maclaurin series, calculus, and a little ingenuity.

||| Exercise 46: Find Maclaurin series for the following functions, using formulas (1) – (8) as appropriate. Try to use (1) as sparingly as possible.

1. $\dfrac{x}{1-x^2}$
2. x^3e^x
3. $\cos^2(3x)$
4. $\sinh(3x)$
5. $\ln(1+x)$
6. $\dfrac{3}{4-5x}$
7. $$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

   Discuss the continuity and differentiability of f at $x = 0$.

□

||| Exercise 47: Assuming convergence as necessary, find the sum

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k+1)}{2^k}.$$

□

||| Exercise 48: Evaluate

$$\sum_{n=1}^{\infty} \left(n \prod_{k=1}^{n} \frac{1}{k} \right).$$

□

In the course of calculating so many derivatives to find Maclaurin series, one's antidifferentiation skills may become a little rusty. So occasionally, there will be a few practice problems to keep them sharp. We'll be doing lots of antiderivatives later, so this is important!

||| Exercise 49: Find the following antiderivatives:

1. $\displaystyle\int \sin^2(3x)\,dx.$
2. $\displaystyle\int \frac{3}{x^2-4}\,dx.$
3. $\displaystyle\int \frac{\ln x}{\sqrt{x}}\,dx.$
4. $\displaystyle\int e^x \tanh x\,dx.$

□

6 Eulerian Dreamtime

It's time for a rather surreal interlude. I invite you to withhold disbelief as you read on. We'll deconstruct with some Freudian analysis after the fact. Feel free to omit on a first reading.

Consider the infinite polynomial

$$P(x) = \cdots (x + 2\pi)(x + \pi)x(x - \pi)(x - 2\pi) \cdots$$

Clearly, $P(x) = 0$ only when $x = n\pi$, $n \in \mathbb{Z}$. Recall that if you know the zeroes of a polynomial, you can reconstruct it up to a constant multiple; for example, if the roots of a quadratic are 3 and e, you know the quadratic looks like

$$k(x - 3)(x - e).$$

Thus, since the zeroes of $\sin x$ are *also* $x = n\pi$, $n \in \mathbb{Z}$, $P(x)$ must be a multiple of $\sin x$.

Now we want to ultimately compare $P(x)$ to the Maclaurin series for $\sin x$. In order to do this, divide each factor $x - n\pi$ of $P(x)$ by the constant $-n\pi$ (except, of course, for the $x - 0\pi = x$ term). This gives

$$Q(x) = \cdots \left(1 + \frac{x}{2\pi}\right)\left(1 + \frac{x}{\pi}\right) x \left(1 - \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right) \cdots .$$

Since the factors occur in conjugate pairs, we may multiply to obtain

$$Q(x) = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right) \cdots ,$$

or in more conventional notation,

$$Q(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

It should be clear that we've got the right multiple of $P(x)$, since the coefficient of the x term of this infinite product is 1. Now let's look at the coefficient of the x^3 term. We must simply choose all terms in the binomials to be 1 *except* for one of the terms, which would be $1 - x^2/n^2\pi^2$ for some n. Doing this in all possible ways gives the term

$$x\left(-\frac{x^2}{1^2\pi^2} - \frac{x^2}{2^2\pi^2} - \frac{x^2}{3^2\pi^2} - \cdots\right) = -\frac{x^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

But we know that the coefficient of x^3 in the Maclaurin expansion for $\sin x$ is $-1/6$, so we conclude that

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6}.$$

Rewriting results in

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

||| Exercise 50: We will use similar ideas to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

1. By considering the x^5 term in the Maclaurin expansion for $\sin x$, find

   $$\sum_{m>n=1}^{\infty} \frac{1}{m^2n^2},$$

   where this sum is an abbreviation:

   $$\sum_{m>n=1}^{\infty} \frac{1}{m^2n^2} = \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{m^2n^2}.$$

2. Write $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ in terms of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{m>n=1}^{\infty} \frac{1}{m^2n^2}$.
3. Use previous results to reach the desired conclusion.

□

||| Exercise 51: Find

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2},$$

that is, the sum of the *odd* reciprocal squares. □

You can probably create many more such problems for yourself.

Why does this all work? Go look it up.

7 Error Analysis

Now we address an important issue: now that we have found Maclaurin series, how good are they as approximations? It turns out that this question does not always have an obvious answer. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It should not be hard to verify that

$$f'(x) = \frac{2}{x^3}e^{-1/x^2}, \quad f''(x) = \frac{2(2-3x^2)}{x^6}e^{-1/x^2}, \quad f'''(x) = \frac{4(6x^4-9x^2+2)}{x^9}e^{-1/x^2}.$$

As x goes to 0, each of these derivatives goes to 0 as well. So do all higher-order derivatives. Our conclusion? The Maclaurin series for this function is 0!

What just happened? It turns out we chose a very *bad* function to approximate using a Maclaurin series. If we zoom in to an interval of $(-0.04, 0.04)$ in the domain of this function, our range is approximately $(0, 3.5 \times 10^{-272})$! Thus, $f(x)$ is *very* 0 near $x = 0$, as perhaps can be deduced from the form of the function.

But at least we have a *little* convergence, since the Maclaurin series converges to 0 at $x = 0$. But admittedly this is a *very* complicated way of calculating 0.

So we have added one more layer of intrigue, for not only do we wish to calculate how good our approximations are, but we would *also* like to decide what for what range of values they *are* good approximations.

It turns out that we really know everything we need in order to make such approximations – since we know something about first-order approximations; that is, tangent lines. This is what some texts colorfully call *the speed-limit law.* Briefly, if we know that our speed (not velocity!) is always less than V, then in time t, our displacement cannot exceed Vt in magnitude.

Recalling that displacement is the integral of velocity, we state this law in mathematical terms as follows. Suppose $x > 0$ for specificity. Also, suppose that for $t \in [0, x]$, $|f'(t)| \leq V$; that is, the speed never exceeds V:

$$-V \leq f'(t) \leq V.$$

Then

$$\int_0^x -V\,dt \leq \int_0^x f'(t)\,dt \leq \int_0^x V\,dt.$$

Evaluating these integrals gives us

$$-Vx \leq f(x) - f(0) \leq Vx,$$

so that
$$|f(x) - f(0)| \le V|x|.$$

We may examine this statement in terms of, say, $f(x) = \sin x$. Clearly, we may use $V = 1$ here, so that we conclude that
$$|\sin x - \sin 0| = |\sin x| \le |x|.$$

So if x is small – say, $x = 0.001$, we know that $|\sin 0.001| \le 0.001$. In fact,
$$\sin 0.001 \approx 0.0009999998333333, \tag{9}$$

so that our approximation is remarkably good. If x is a bit larger – say $x = 1/2$ – we know that
$$\sin(1/2) \le 1/2,$$

but it turns that we are accurate to just one decimal place: $\sin(1/2) \approx 0.48$. And of course, for large x – say $x = 2$, the accuracy estimate is meaningless, since we already know that $|\sin x| \le 1$ for all x.

Of course there is no need to stop at the speed-limit law. We have an estimate of maximum displacement if we have a limit on our speed. It is possible to perform precisely the same calculation if we have a limit on the magnitude of our acceleration as well. Can you think of what it might be? Assuming you begin at time $t = 0$ at $x = 0$ with initial velocity $v = 0$, can you determine the maximum displacement given that your acceleration is always less that some positive number A in magnitude?

||| Exercise 52: Suppose $x > 0$ is given, and assume that for $t \in [0, x]$, we have $|f''(t)| \le K$ for some positive K. Using the same idea as above for the speed-limit law, estimate the error of the first-order approximation to $f(x)$. □

You should have obtained something like
$$|f(x) - (f(0) + f'(0)x)| \le \frac{1}{2}Kx^2.$$

I hope you can see the pattern, and see *why.* If we use one more term in our expansion (that is, the $f'(0)x$ term), our accuracy is now quadratic with x. This means that if x is small, we are more accurate.

Now let's consider this result with our example $f(x) = \sin x$ for $x > 0$. Since we know $f''(x) = -\sin x$, we may use $K = 1$ to give the approximation
$$|\sin x - x| \le \frac{1}{2}x^2.$$

But we already know that $\sin x < x$, so that we may rewrite this as
$$x - \sin x \le \frac{1}{2}x^2.$$

Thus, $\sin 0.001$ is closer to 0.001 than 5×10^{-7}, and $\sin(1/2)$ is closer to $1/2$ than $1/8$.

Hopefully at this point, the pattern for these error estimates is evident, so that we may write a general result without needing to perform all the calculations. To do so, we employ the usual notation

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

which represents the nth order approximation to $f(x)$. This is a compact notation, so a few reminders are needed. First, we have $0! = 1$. Second, $f^{(n)}$ means the nth derivative of f. In particular, the 0th derivative (that is, taking *no* derivatives) is simply f itself, so that

$$f^{(0)}(0) = f(0).$$

Then the results of our work show that:

THEOREM 4: (**Polynomial Estimation Theorem**) Let a function f be given which is defined on an interval I, and let $n > 0$ be given. Then for all $x \in I$, if $K > 0$ is such that $\left|f^{(n+1)}(t)\right| \leq K$ for $t \in I$, we have

$$|f(x) - P_n(x)| \leq \frac{K}{(n+1)!}|x|^{n+1}.$$

(**Exact Form**) Moreover, for all $x \in I$, there is a point $c \in I$, depending on x, such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Let's take a moment to explain the presence of "I" in this theorem. In our previous examples, we used $I = [0, x]$ since we assumed that x was positive. This more general case allows us to consider x negative by using the interval $I = [x, 0]$ in this case. The Exact Form is stated here for completeness, but in practice, the determination of c is not a manageable task

||| EXERCISE 53: Use the previous result with $n = 2$, and discuss the corresponding approximations to $\sin 0.001$ and $\sin(1/2)$. □

It is worthwhile to take a moment to think about what Theorem 4 means. Evidently, we can get simple estimates on how close $P_n(x)$ is to $f(x)$ by knowing something about higher-order derivatives of $f(x)$.

||| EXERCISE 54:

1. Find the fourth-order approximation to $\cos 1$.

2. What is the best estimate for the error for this approximation using Theorem 4?

3. Using Theorem 4, what is the lowest-order approximation which will insure an error no greater than 10^{-10}?

□

||| Exercise 55: Repeat the previous exercise, except with e^1. □

||| Exercise 56: Consider the function $f(x) = \sqrt{x+1}$.

1. Find $P_2(x)$; that is, the second-order polynomial approximation to $f(x)$.
2. Estimate the error of the approximation when used to find $\sqrt{2}$.
3. Using $\sqrt{2} \approx 1.414$, verify that your approximation is within this error.

□

||| Exercise 57: Consider the function $f(x) = \dfrac{1}{1+2x}$.

1. Give the Maclaurin series for this function, and indicate for which x this series converges.
2. Find the second-order polynomial approximation to $f(x)$, and use this to approximate $f(1/4)$.
3. Calculate $f(1/4)$ exactly. Show that your approximation is within the error predicted by the Polynomial Estimation Theorem.

□

Before moving on, a comment on finding the "K" in Theorem 4 is in order. In the exercises above, K was independent of n. For example, given $f(x) = \sin x$, we may choose $K = 1$ regardless of the order of the approximation, since the derivatives of $f(x)$ cycle among $\cos x$, $-\sin x$, $-\cos x$, and $\sin x$, and none of these ever attains a value larger than 1.

It is usually the case that the calculation of K *does* depend on n, as in the following exercise.

||| Exercise 58: Let $f(x) = \sin(2x)$. Find the fourth-order approximation to $f(0.01)$, and estimate the error using the Polynomial Estimation Theorem. Then verify that your approximation is within this error. □

||| Exercise 59: Prove or provide a counterexample: Suppose that $f(x)$ and $g(x)$ have Maclaurin series which converge for all real x. Let $P(x)$ be the second-order approximation

to $f(x)$, and let $Q(x)$ be the second-order approximation to $g(x)$. Then $P(x)Q(x)$ is the fourth-order approximation to $f(x)g(x)$. $\square$

||| EXERCISE 60: For what positive integers n does the Maclaurin series for $f(x) = |x|^n$ exist and converge for all real x? $\square$

||| EXERCISE 61: Suppose a function f has the property that the polynomial approximations P_n satisfy $P_n(1) = n$ for all $n \geq 0$. Find f. $\square$

||| EXERCISE 62: Suppose you know that a function f has a Maclaurin series. Further, you know that $f^{(n)}(x)$ exists for all real numbers and for each $n \geq 0$. Finally, you know that

$$\left|f^{(n)}(x)\right| \leq n^2$$

for all real x and $n \geq 1$. What can you say about the interval of convergence of the Maclaurin series for f? $\square$

||| EXERCISE 63: Suppose that the function f, on some interval I, satisfies

$$0 \leq f^{(n)}(x) \leq \frac{1}{2^n}$$

for $n \leq 3$. Approximate the error when $P_2(x)$ is used to approximate the Maclaurin series for $f(f(x))$ at $x = a \in I$. $\square$

8 Intervals of Convergence

We're now ready to answer – at least in part – the question of which values of x a given Maclaurin series converges for. Many issues arise, so we'll take things one step at a time.

First, we'll take a look at the Maclaurin series for $\sin x$. We know that

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad |\sin x - P_n(x)| \le \frac{|x|^{n+1}}{(n+1)!},$$

since we may certainly use $K = 1$ in Theorem 4.

Examining the estimate for $P_n(x)$, it is evident that if

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0,$$

then we should have

$$\lim_{n\to\infty} P_n(x) = \sin x;$$

that is, the approximations $P_n(x)$ converge to $\sin x$ *for all* x. This would be good news indeed – that the Macluarin series for $\sin x$ converges for all x – since for many series (geometric, for example), this is most definitely not the case. Take a moment to convince yourself that this limit is indeed 0. Can you prove it?

To avoid carrying around an "$n + 1$" everywhere, we instead establish that

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0.$$

Please follow carefully; the idea in this proof is central to much of what we do in considering convergence. The idea is essentially this: we favorably compare the sequence $|x|^n/n!$ to a geometric sequence. Here we go!

For a given x, first choose $M > 0$ such that $|x| < M$. The idea is to compare $\dfrac{|x|^n}{n!}$ to $\left(\dfrac{|x|}{M}\right)^n$. Since this is a geometric sequence, and since we know a *lot* about geometric sequences, this could conceivably be helpful. If you'd like, take a moment to try and complete the proof yourself. If not, read on....

First, suppose that $n > M$. Since we are taking a limit as n tends toward infinity, we haven't in any way compromised our goal. Note that

$$n! = n(n-1)\cdots(M+1)M(M-1)\cdots 1.$$

The first $n - M$ terms of the product are each larger than M, so that

$$n! > M^{n-M}M!.$$

This implies that

$$\frac{1}{n!} < \frac{1}{M^{n-M}M!},$$

which in turn results in

$$0 < \frac{|x|^n}{n!} < \frac{|x|^n}{M^{n-M}M!} = \left(\frac{|x|}{M}\right)^n \frac{M^M}{M!}.$$

And this is it! Recall that once we have chosen x, then M is determined, so that $M^M/M!$ is a constant. So as $n \to \infty$, the rightmost expression must go to 0 since $|x|/M < 1$. Thus, our old friend the Squeeze Theorem gives us the result we want:

$$\lim_{n\to\infty} \frac{|x|^n}{n!} = 0.$$

Further, we conclude that the Maclaurin series for $\sin x$ must converge to $\sin x$ for every value of x.

First, take a moment to assure yourself that you understand this argument *clearly.* Next, reflect on what made this proof go so nicely. In Theorem 4, we need to determine K for *each* value of n – and, in general, K does indeed depend on n (as we saw in a previous exercise). If K just depends on x and *not* on n, we may rewrite the inequality of Theorem 4 as

$$|\sin x - P_n(x)| \le \frac{K(x)}{(n+1)!}|x|^{n+1},$$

indicating that K is independent of n. Please be aware that the case when K is independent of x as well (as in our previous example involving $\sin x$) is *very* rare. So don't count on that happening too often.... Now since we are determining convergence as n tends toward infinity, the $K(x)$ term will be constant in our calculations, so that the only limit relevant to convergence is

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!}.$$

But of course we have just determined that this is 0. Thus, if we can establish that K is independent of n for every n, then the corresponding Maclaurin series converges for every value of x.

||| Exercise 64: Of the basic examples previously considered:

$$\sin x, \quad \sinh x, \quad \cos x, \quad \cosh x, \quad e^x, \quad \frac{1}{1+x}, \quad \frac{1}{1-x},$$

which converge for every value of x using this criterion? □

Now let's take a detailed look at another example of the dependence of K on n. One fairly simple example is

$$f(x) = \frac{1}{1-x},$$

a friendly geometric series. We first establish that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

Now you can likely take a few derivatives and try to find a pattern, but we're going to go all the way here and actually prove that such a "guess" is valid using mathematical induction.

We first look at the base case. Here, we may take $n = 0$, and so indeed

$$f(x) = f^{(0)}(x) = \frac{0!}{(1-x)^1} = \frac{1}{1-x}.$$

Proceeding to the induction step, assume that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

Then

$$f^{(n+1)}(x) = \frac{d}{dx} n!(1-x)^{-(n+1)} = -(n+1)n!(1-x)^{-(n+1)-1}(-1) = \frac{(n+1)!}{(1-x)^{n+2}}.$$

As this establishes our assumption with n replaced by $n+1$, we see by the principle of mathematical induction that our conjecture is true for all $n \geq 0$.

Now what does Theorem 4 say about the convergence of the Maclaurin series for $f(x)$? Let's assume that $0 < x < 1$ (recall that this is a geometric series), and that n is given. Then K_{n+1} (where the subscript denotes the dependence on n in this case) is the maximum value of $f^{(n+1)}(t)$ for $t \in [0, x]$. It should be clear that $f^{(n+1)}$ is increasing – note that $f^{(n+2)}$ is strictly positive. Thus, the maximum occurs at the right end point of the interval $[0, x]$, so that

$$K_{n+1} = \frac{(n+1)!}{(1-x)^{n+2}}.$$

Hence,

$$\left|\frac{1}{1-x} - P_n(x)\right| \leq \frac{(n+1)!\, x^{n+1}}{(n+1)!(1-x)^{n+2}} = \frac{x^{n+1}}{(1-x)^{n+2}} = \frac{1}{1-x}\left(\frac{x}{1-x}\right)^{n+1}.$$

Recall that $0 < x < 1$. We see that the estimates go to 0 precisely when x is positive and $\dfrac{x}{1-x} < 1$. But this occurs only when $x < 1/2$! When $x = 2/3$, for example, it is easy to see that

$$\left|\frac{1}{1-x} - P_n(x)\right| \leq 3 \cdot 2^{n+1},$$

which is clearly *not* an encouraging estimate for the error. A moment's thought suggests that the asymptotic behavior of the derivatives of f near 1 plays a significant role in these poor estimates. But we hope the moral of the story is clear: even with the most basic of series – the geometric series – Theorem 4 does not suffice to determine the interval of convergence.

Well, then what? Let's first take a look at a simple example:

$$\sum_{n=0}^{\infty} nx^n. \tag{10}$$

Now in this case, you don't know what function produced this series (although we have hinted at it earlier). But regardless of how it arose, we may still ask: for what values of x does it converge? Clearly, if $x > 1$, the series diverges. But for other values – like $x = 1/5$, it seems plausible that the powers of x could drive the series to converge despite the coefficient of n. For x near 1, though, it might get a little tricky.

We motivate what comes next from considering – again – geometric series. If a series is *essentially* geometric, then it must converge. What does it mean for a series to be essentially geometric? Let's look for a moment at

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) \left(\frac{2}{3}\right)^n.$$

Now this series is certainly *not* geometric. But if you think about what happens as n gets larger, you can see that this series behaves very much like a geometric series, since the "$1/n^2$" term becomes insignificant when added to 1.

Now how can we make this notion precise? Quite simply, the ratio between any two terms in a geometric series is constant. So in considering the series (10), we look at the ratios

$$\frac{(n+1)x^{n+1}}{nx^n}$$

as n tends toward infinity. These ratios – if (10) is to behave like a geometric series – should have a limit which is less than 1 in magnitude. In other words, we should have

$$\lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| < 1.$$

But evaluating this limit gives $|x| < 1$, and so we conclude that if $|x| < 1$, then (10) converges, as it is "essentially geometric;" that is, if you go out far enough, it behaves like a geometric series.

It should be just as clear, hopefully, that if $|x| > 1$, the series diverges. And in this case, if $|x| = 1$, the series diverges. But in general, these boundary points of the interval of convergence are rather more problematic. The reason is this. We know that if $|r| = 1$, then the geometric series

$$\sum_{n=0}^{\infty} ar^n$$

converges if and only if $a = 0$. This should be obvious.

But suppose a can vary with n, as with the series

$$\sum_{n=0}^{\infty} a_n r^n.$$

Then this series can converge when $|r| = 1$ if the a_n go to 0 quickly enough. For example, if $a_n = e^{-n}$, the series converges when $|r| = 1$ – take a moment to see why! And as it turns out, such a series may converge when $r = 1$, but *diverge* when $r = -1$, or vice versa. Strange behavior! So determining convergence at the boundary of the interval of convergence must be done on a case-by-case basis.

So let's summarize the result of our work so far, and then give you some practice! We note that the theorem is stated in terms of "b_n." For series, as we have just encountered, simply put $b_n = a_n x^n$. But the theorem is applicable to ordinary series as well, as you will encounter in the exercises.

Theorem 5: (**Ratio Test**) Let the series

$$S = \sum_{n=0}^{\infty} b_n$$

be given, with $b_n \neq 0$ for $n \in \mathbb{N}$. If

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L$$

exists, then S converges if $L < 1$ and diverges if $L > 1$. If $L = 1$, no conclusion can be drawn.

Let's first practice using the Ratio Test to determine the convergence or divergence of several different series.

||| Exercise 65: Discuss the convergence of

$$\sum_{k=1}^{\infty} \frac{e^{2k}}{(2e)^k}.$$

□

||| Exercise 66: Discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{n^3 3^n}{n!}.$$

□

||| EXERCISE 67: Discuss the convergence of

$$\sum_{m=1}^{\infty} \frac{(2m)!}{(m!)^2}.$$

□

||| EXERCISE 68: Discuss the convergence of

$$\sum_{k=1}^{\infty} \frac{\ln k!}{e^k}.$$

□

Before moving on to the next set of exercise, a remark about the limit L in the statement of the Ratio Test is in order. Given a series

$$\sum_{n=0}^{\infty} a_n x^n,$$

we can establish convergence if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right|$$

exists and is less than 1. Now suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 0.$$

Then the given series converges when

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \cdot x \right| = L|x| < 1,$$

or equivalently, when $|x| < 1/L$. In other words, not only does the limit obtained by the Ratio Test applied to the coefficients of the Maclaurin series determine whether the series converges, it also directly reveals the interval of convergence. This is precisely why the Ratio Test is of such importance.

In attempting the following exercises, decide as best as you can what happens at the endpoints of the interval of convergence. Try to justify why you think a series converges or diverges at the endpoints of an interval. Don't worry if you can't, though – we'll be spending some time looking at several of these cases later. The idea is to get you thinking about various different types of series.

||| EXERCISE 69: Find the interval of convergence for the series

$$\sum_{n=0}^{\infty} e^n x^n$$

in two ways. First, use the Ratio Test. Second, rewrite it as a geometric series. Finally, determine the sum of the series when it does converge. □

||| EXERCISE 70: Find the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

□

||| EXERCISE 71: Let $k > 0$ be a given positive number. Find the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^k}.$$

□

||| EXERCISE 72: Find the interval of convergence for the series

$$\sum_{n=0}^{\infty} \sin\left(2^{-n}\right) x^n.$$

□

||| EXERCISE 73: Find the interval of convergence for the series

$$\sum_{n=0}^{\infty} (x+3)^n.$$

□

||| EXERCISE 74: Find the interval of convergence for the series

$$\sum_{n=0}^{\infty} (2n+1)(x-4)^n.$$

□

||| EXERCISE 75: Determine the interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}.$$

□

||| Exercise 76: Find the interval of convergence for the series

$$\sum_{n=0}^{\infty} n!\, x^n.$$

□

||| Exercise 77: Determine the interval of convergence for the series

$$\sum_{n=0}^{\infty} (x-n)^n.$$

□

||| Exercise 78: Consider the series

$$1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{27} + \cdots + \frac{1}{2^{2n}} + \frac{1}{3^{2n+1}} + \cdots$$

Show that this series converges, even though the Ratio Test fails to apply. Find the sum of the series. □

||| Exercise 79: Determine what relationship must obtain between positive integers p, q, and r in order for the series

$$\sum_{n=1}^{\infty} \frac{(pn)!(qn)!}{(rn)!}$$

to converge. □

9 Alternating Series

You likely noticed that there was a lot going on in the previous exercises: hard-to-determine behavior at the endpoints of intervals of convergence, "$(x+3)^n$" terms rather than just "x^n" terms, and a case where the Ratio Test did not apply. In other words, gone are the days when we are merely content to take a few derivatives, write down a Maclaurin series, and call it an afternoon.

It will take us some time to address all the issues that have arisen. *Carpe diem!*

The simplest issue concerns

$$\sum_{n=0}^{\infty} e^n x^n.$$

This, as we saw, is a geometric series with interval of convergence $(-e^{-1}, e^{-1})$. Why doesn't the series converge at the endpoints of this interval?

Just to the left of $1/e$, each term in the series is a power of a number less than 1, and so the terms get smaller and smaller, eventually going to 0. However, right *at* $1/e$, each term is 1, and so the sum just keeps getting larger and larger. It should be clear that *if* the sum is to converge, the individual terms *must* go to 0.

This is made explicit in the following:

THEOREM 6: (**Disappearing Sequence Test**) Suppose a series

$$\sum_{n=0}^{\infty} a_n$$

is given. Then the series diverges if either $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$.

Note that many texts call this the "nth" term test – but since it is not good practice to leave unidentified variables (such as "n") loitering around, we opt for a slightly more colorful name.

||| EXERCISE 80: In the previous example, it seems that when $x = -1/e$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 0 + 0 + 0 + \cdots \\ &= 0.\end{aligned}$$

This appears to violate the Disappearing Sequence Test. Explain. □

The Disappearing Sequence Test is a simple and useful test – so don't underestimate it. One often has the tendency to look at more complicated tests for convergence first, when occasionally the simplest will suffice.

The next example we will consider is the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Recall that a harmonic sequence is defined to be the termwise reciprocals of an arithmetic sequence, hence the name.

Let's recall the problem. For the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n},$$

the interval of convergence is $[-1, 1)$. But wait, you say! When $x = 1$, we have the sum

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

and by the Disappearing Sequence Test, this series converges!

Let this be a learning experience – and learn it now, once and for all. What the Disappearing Sequence Test says is that **IF** $\lim_{n\to\infty} a_n$ diverges or is not 0, **THEN** the series diverges. This is logically equivalent to the statement that **IF** the series converges, **THEN** it follows that $\lim_{n\to\infty} a_n = 0$. (This is called the *contrapositive* statement in mathematical logic.) The practical importance of this remark is that if you know $\lim_{n\to\infty} a_n = 0$, you can conlude *nothing* about the series. It might converge, or it might diverge. Take a moment to think about this, and make sure it is very clear in your mind.

||| EXERCISE 81: Show that

$$\sum_{n=0}^{\infty} \sin(2^{-n})x^n$$

diverges at the endpoints of its interval of convergence, $(-2, 2)$. □

We'll address this particular issue later when we discuss the Integral Test, and tackle the easier endpoint, $x = -1$. In this case, we're looking at the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots,$$

which is called the **alternating harmonic series**. Can you think about why this converges? We'll look at this example in some detail, since it essentially gives a proof for the more general result.

As you know, this series converges. In this case, we also know that the terms in the sequence converge to 0. But *this is not why the series converges.* As mentioned just a moment ago, the Disappearing Sequence Test does *not* apply in this case. The series converges for other reasons, as we will now see.

First, let's rewrite this sum as

$$\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n},$$

so that $a_n = 1/n$ for $n \geq 1$, and let S_n be the nth partial sum. Now observe that this sum may be written

$$1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots .$$

From this, it is clear that

$$a_1 = S_1 \geq S_3 \geq S_5 \geq \cdots \geq S_{2n+1} \geq \cdots$$

In order for this to occur, we use the fact that a_n is a decreasing sequence; that is, $a_n \geq a_{n+1}$ for all $n \geq 1$. Take a moment to think about why this is important in this case – it implies that each subtraction in parentheses yields a positive result.

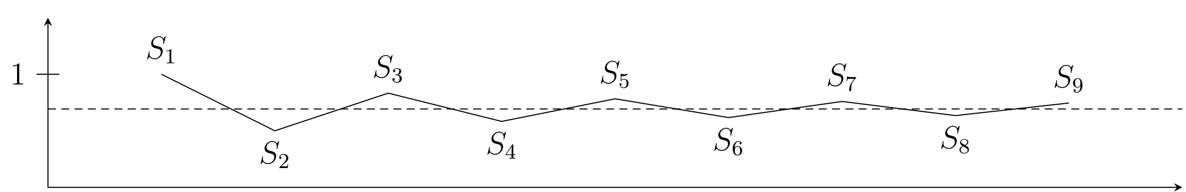
Now let's write the series in a slightly different way:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots$$

This clearly implies that

$$S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_{2n} \leq \cdots$$

Again, the assumption that a_n is decreasing is important here. To help make the two series of inequalities a little more concrete, below is a diagram which shows the behavior of the partial sums.



But we saw earlier that the sum of the series is bounded above by $a_1 = S_1$. Thus, the even partial sums keep increasing, but are bounded above by S_1. Now recall the Monotone Convergence Theorem: if a sequence of real numbers is increasing and bounded above, it must converge. Thus, the even partial sums converge to some limit; let's call it S_E.

Now take a brief moment and convince yourself that the odd partial sums are bounded *below* by S_2, and so the odd partial sums are a decreasing sequence bounded below. The Monotone Convergence Theorem also applies in this case as well, so that the odd partial sums converge to some limit, say S_O.

Are the limits S_E and S_O the same? From the definition of the partial sums, it should be clear that

$$|S_{n+1} - S_n| = a_{n+1}.$$

And of course, if $n+1$ is odd, then n is even, and if $n+1$ is even, then n is odd. Thus, as long as $\lim_{n\to\infty} a_n = 0$, the even and odd partial sums get closer and closer to each other. This is the case here, since clearly $\lim_{n\to\infty} 1/n = 0$. Therefore we have $S_E = S_O$, and so the series converges.

It turns out that all of our hard work *also* gives us an estimate of the accuracy of the partial sums, practically for free. We noticed that the partial sums are bounded above by $a_1 = S_1$. Let S denote the sum of this series. Then, for example, we may write

$$S - S_6 = \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

But note that the right-hand side of this equation is *itself* an alternating series, and hence bounded above by the first term, which is $1/7$. Thus,

$$S - S_6 \le \frac{1}{7}.$$

And of course, this is true anywhere we wish to stop along the way. We call $S - S_6$ a **tail** of the series; in general, a tail of a series is obtained by leaving off any number of the initial terms. A tail is also called a **remainder**, hopefully for obvious reasons.

For odd partial sums, the equations look like

$$S_7 - S = \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \frac{1}{11} + \cdots ,$$

but the idea is the same.

So as long as we believe in the Monotone Convergence Theorem, we've got ourselves a proof of the Alternating Series Test:

Theorem 7: (**Alternating Series Test**) Suppose that a_n is a decreasing sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$. Then

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converges to a limit S, and

$$|S - S_n| \le a_{n+1}$$

for all $n \ge 1$. Moreover, S is larger than every even partial sum, and smaller than every odd partial sum.

A few final remarks are in order. First, note that in our proof, we grouped the terms in pairs, but we *kept them in the same order* when we considered the sum. The series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots \tag{11}$$

is definitely *not* the alternating harmonic series. Think about how you add the series, and note that you are, in some sense, using up the even terms "twice as fast" as the odd terms. Now it turns out that this sequence *does* converge, but *not* to the same limit as the alternating harmonic sequence! More on this later.

Second, when considering convergence of series, it's really only the tails which are important. Thus, we might say that the series

$$\pi - e + 42 - 7 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges by the Alternating Series Test, since we can *always* sum a finite number of terms. It's only when we're adding an *infinite* number of terms – which is always the case when adding terms in a tail – that we need to be especially cautious.

And finally, note that whether the sum begins at $n = 0$ or $n = 1$ is irrelevant, as long as the terms of the resulting series alternate in sign.

||| Exercise 82: Consider the sequence $a_n = 1 + 2^{-n}$, $n \ge 0$. This is clearly a decreasing sequence of positive numbers. Consider the series

$$\sum_{n=0}^{\infty}(-1)^n a_n.$$

Show that the sequence of odd partial sums converges, as does the sequence of even partial sums, but *not to the same limit.* Does this violate the Alternating Series Test? □

||| Exercise 83: Consider the Maclaurin series for $\sin x$. Which gives a sharper bound on

the error, the Polynomial Estimation Theorem, or the Alternating Series Test? □

||| Exercise 84: Consider the alternating series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n},$$

and let S_n be the nth partial sum

$$S_n = \sum_{k=0}^{n} \frac{(-1)^k}{2^k}.$$

1. Estimate the error $|S - S_n|$ using the Alternating Series Test.
2. Calculate the actual error $|S - S_n|$.

□

One more definition.

Definition 8: An alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n,$$

where $a_n \geq 0$ for all $n \geq 1$, is said to be **absolutely convergent** (**conditionally convergent**) if it converges and

$$\sum_{n=1}^{\infty} a_n$$

also converges (does not converge).

Of course we just encountered the alternating harmonic series, which is conditionally convergent. The consideration of absolutely and conditionally convergent alternating series comes up not infrequently when finding intervals of convergence.

The distinction between conditionally and absolutely convergent alternating series is an interesting one. We'll illustrate with the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Now it happens that the sum of the positive terms diverges, as does the sum of the negative terms – and this will happen for any conditionally convergent alternating series. The reason is fairly simple. We know the alternating harmonic series converges. If the sum of the negative terms converges, we can add the convergent series

$$0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \frac{1}{6} + \cdots$$

to the alternating harmonic series, resulting in another convergent series: the positive terms. Thus the sum of these two convergent series,

$$\left(1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \cdots\right) + \left(0 + \frac{1}{2} + 0 + \frac{1}{4} + 0 + \cdots\right),$$

must *also* converge. This contradicts the fact that the alternating harmonic series is conditionally convergent.

In particular, this means that we cannot simply "rearrange" terms of the alternating harmonic series, as in (11), with any assurance that the limit is the same. In fact, it is possible to rearrange the terms of the alternating harmonic series to obtain *any* number you want. And if you ask me nicely, I'll even show you.

||| EXERCISE 85: Show that if an alternating geometric series converges, it converges absolutely. □

||| EXERCISE 86: Consider an alternating geometric series

$$\sum_{n=0}^{\infty} ar^n,$$

where $a > 0$ and $-1 < r < 0$. Verify that the terms of an absolutely convergent series may be rearranged by first adding the positive terms in the series, then the negative terms, and then showing that their sum is indeed the sum of the series. □

||| EXERCISE 87: Discuss the convergence of

$$\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\ln n}.$$

□

||| EXERCISE 88: Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \tan\left(\frac{(2n-1)\pi}{4}\right).$$

□

||| EXERCISE 89: Discuss the convergence of the series

$$\sum_{n=0}^{\infty}(-1)^n \cos(\sin(\arctan n)).$$

□

||| EXERCISE 90: Suppose that a_n is a sequence of positive terms with the following properties:

1. $a_n^2 > a_{n+1}^2$ for all $n \geq 0$;
2. $\lim_{n\to\infty} a_n^2 = 0$.

For each of the following statements, prove or provide a counterexample:

1. If $\sum_{n=0}^{\infty} a_n^2$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
2. If $\sum_{n=0}^{\infty}(-1)^n a_n^2$ converges, then $\sum_{n=0}^{\infty}(-1)^n a_n$ converges.

□

||| EXERCISE 91: Let a_n be a decreasing sequence of positive numbers. Suppose that the alternating series

$$\sum_{n=0}^{\infty}(-1)^n a_n$$

is conditionally convergent.

Now define $b_n = (-1)^n a_n$ for $n \geq 0$.

1. Show that $\sum_{n=0}^{\infty} b_{3n}$ converges.
2. Is $\sum_{n=0}^{\infty} b_{3n}$ conditionally convergent or absolutely convergent? Justify your answer.

□

Since the Integral Test is coming up, it's time for a little practice.

||| Exercise 92: Antidifferentiate the following:

1. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx.$
2. $\displaystyle\int x \arcsin x\,dx.$
3. $\displaystyle\int e^x \sin x\,dx.$

□

10 The Integral Test

Earlier, we showed that the harmonic series

$$H = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by cleverly comparing it to a series easily shown to diverge. Also, recall that such series regularly arose when considering the endpoints of intervals of convergence. Now, we investigate another method to show the divergence of the harmonic series – but a method with much broader application. Not that being clever doesn't have broad application....

I'll start you off with a hint. After that, you're welcome to try to prove it yourself – or, if you're not so inclined, read on. The idea is this: H is a sum which approximates an integral.

Now this might not have immediately occurred to you. And why not? Because it's a really *bad* approximation. And not only that, the sum isn't in quite the form we are used to seeing for approximating integrals. What *is* important is that you recall the following two facts, which naturally you remember quite vividly:

1. For decreasing functions, left-hand approximations *overestimate* the integral, and
2. For decreasing functions, right-hand approximations *underestimate* the integral.

Take a moment and think about exactly *what* integral H may be over- or underestimating. We'll first consider

$$\int_1^{\infty} \frac{1}{x}\, dx.$$

What, you may ask, is that "∞" doing as a limit of integration? It's not really all that complicated. This simply means that we're looking at an area under a function which has the x-axis as asymptote. Just as some series converge or diverge, we also say that such integrals converge or diverge – in other words, the accumulated area approaches a limit, or grows without bound (or perhaps diverges in some other way). In particular:

DEFINITION 9: (**Improper Integrals**) We define

$$\int_a^{\infty} f(x)\, dx = \lim_{N\to\infty} \int_a^{N} f(x)\, dx.$$

We say the integral **converges** if this limit exists, and **diverges** if it doesn't.

Of course there really is nothing improper at all about such integrals, but unfortunately, the name has stuck. But you should know what most of the civilized world (or at least the calculus books it produces) calls such integrals. Remember it well, for you will not see such terminology used here very frequently.

In any case, you will no doubt notice the similarity to the definition of a series as a limit of a sequence of partial sums. Take a moment and make sure this makes sense to you.

||| Exercise 93: Determine whether

$$\int_1^\infty \frac{1}{x}\,dx$$

converges or diverges. □

||| Exercise 94: Let a positive number $k > 0$ be given, with $k \neq 1$. Determine whether

$$\int_1^\infty \frac{1}{x^k}\,dx$$

converges or diverges. □

||| Exercise 95: Determine whether

$$\int_0^1 \frac{1}{x}\,dx$$

converges or diverges. □

If you didn't get the last exercise, don't worry. We'll get to that later. Right now, we've got other fish to fry, as it were.

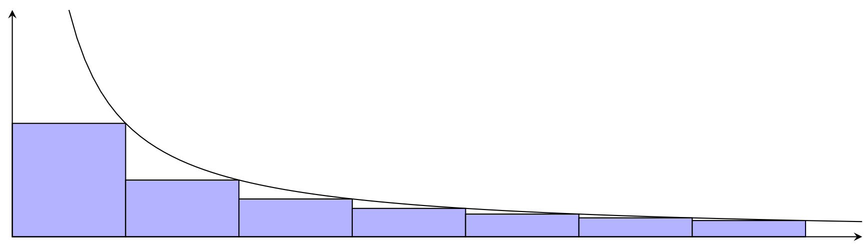
Back to H. Let's consider the partial sums

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

We'll consider H as both a right-hand approximation and a left-hand approximation – since at the moment it may be difficult to decide which is more relevant. We see that H_n is a right-hand approximation to

$$\int_0^n \frac{1}{x}\,dx,$$

as we see in the drawing below with $n = 7$.



But this is a little problematic (see the previous exercise), since there is a vertical asymptote at $x = 0$. But convergence doesn't depend on the first one (or several) term(s), so let's rearrange slightly and say that

$$H_n - 1 < \int_1^n \frac{1}{x}\,dx.$$

There, that's better! And it won't hurt our argument at all. But do notice the inequality! Remember, right-hand approximations underestimate areas for decreasing functions. But we just showed that

$$\lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx$$

diverges – and looking back at our computation, this limit DNE($+\infty$). So we have no useful information about $\lim_{n\to\infty} H_n$.

But this doesn't mean our work is without merit. For if the integral *did* converge, then H_n would be an increasing sequence bounded above by the value of this integral. Therefore, by the Monotone Convergence Theorem, the series would converge!

We can make yet another conclusion from our (apparently) useless inequality. What would happen if we knew the series diverged? Since the integrals are larger than the partial sums (minus one), then this would mean that the integral

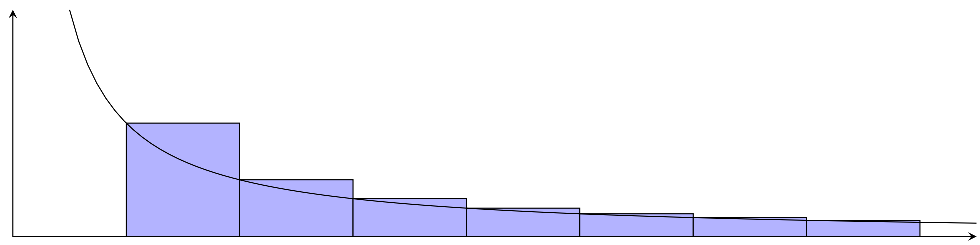
$$\int_1^\infty \frac{1}{x}\,dx$$

diverged as well. But please note: this presumed that we could conclude that the partial sums increased without bound, so we could say that the integrals kept getting larger without bound. This is an important assumption, so make sure you understand its role in the argument!

OK, so looking at right-hand approximations didn't provide enough information to decide whether or not the partial sums converged. So let's look at the left-hand approximations. Without any modification necessary, we see that H_n is a left-hand approximation to

$$\int_1^{n+1} \frac{1}{x}\,dx,$$

shown in the drawing below with $n = 7$.



And since left-hand approximations overestimate, we see that

$$H_n > \int_1^{n+1} \frac{1}{x}\,dx.$$

But now we can use our hard-won knowledge – since we know H_n is *greater* than this integral, and that the corresponding sequence of integrals diverges, we can conclude that the sequence of partial sums also diverges. Thus, H diverges. Finally!

It is worth pointing out that if the integrals did converge, that would provide us no information about H. And if we knew that the partial sums converged, that would mean the integrals converged as well.

And, as expected, a few words about error are in order. Only a few – because we've really done all the work already. Let's suppose a series $A = \sum_{n=1}^{\infty} a_n$ consisting of positive terms converges, and let A_n be the corresponding sequence of partial sums. For specificity, let's assume there is a decreasing function $a(x)$ such that $a_n = a(n)$. Then we may rewrite our previous inequalities as

$$\int_1^{n+1} a(x)\,dx \le A_n \le a_1 + \int_1^{n} a(x)\,dx. \tag{12}$$

A few remarks are in order. We needed to use "$\le$" here since it is possible that over some interval, $a(x)$ is constant. (This didn't occur in considering the harmonic series.) And second, "a_1" is here instead of the "1" previously. But note that this 1 was just the first term in the series, so in general, this term is just a_1. A quick glance back at the drawings should convince you of these inequalities.

Now in taking the limit at n approaches infinity, we have

$$\int_1^{\infty} a(x)\,dx \le A \le a_1 + \int_1^{\infty} a(x)\,dx.$$

To get an estimate on the remainders

$$R_n = \sum_{k=n+1}^{\infty} a_k,$$

we simply note that R_n is nothing more than a right-hand approximation to the integral

$$\int_n^\infty a(x)\,dx.$$

And since right-hand approximations underestimate, we have

$$R_n \leq \int_n^\infty a(x)\,dx$$

for $n \geq 1$.

To summarize:

Theorem 10: (**Integral Test**) Consider the series

$$A = \sum_{n=1}^\infty a_n,$$

with partial sums A_n and remainders R_n, and let a be a continuous function such that $a_n = a(n)$, $n \geq 1$. Then the series and the integral

$$\int_1^\infty a(x)\,dx$$

either both converge, or both diverge. In the event that they both converge, we have

$$\int_1^{n+1} a(x)\,dx \leq A_n \leq a_1 + \int_1^n a(x)\,dx \quad \text{and} \quad R_n \leq \int_n^\infty a(x)\,dx$$

for $n \geq 1$.

Again, a few remarks. We usually begin with $n = 1$ here since in many cases, the Integral Test is used when there is a vertical asymptote at $x = 0$. Please do not try to *memorize* the inequality (12) given in the theorem. I realize that boxes are value-neutral; that is, they are inherently neither good nor evil. However, they may be used in more-or-less evil ways; that is, representing things to merely memorize. Rather than use this perfectly innocent box in such a ghastly way, just recall that the Integral Test was proved by considering right-hand and left-hand approximations to integrals. With this in mind, you should be able to draw a quick sketch and derive from that any inequality you might need.

||| Exercise 96: Decide for which $k > 0$ the series

$$\sum_{n=1}^\infty \frac{1}{n^k} \tag{13}$$

converges. $\square$

You should also be aware that the result of the previous exercise is often called the **p-test** in calculus texts – so called because the variable p is used rather than k. This is a historical artifact, and we shall not dignify this easy consequence of the Integral Test with a box. Again, the dummy variable in the name of the theorem gives one pause.

This does not, however, mean that the result is not useful – as many series can be compared to series given by (13).

||| Exercise 97: Decide whether or not

$$\sum_{n=2}^{\infty} \frac{3n}{\sqrt{n^3 - 1}}$$

converges. $\square$

Before getting to more practice, it's time to look back at a previous exercise: determining whether or not

$$\int_0^1 \frac{1}{x}\,dx$$

converged. Perhaps you've already figured out that you need to examine

$$\lim_{a \to 0^+} \int_a^1 \frac{1}{x}\,dx$$

By drawing a picture, the nature of the limit you need to take to evaluate integrals like this should always be clear.

But there is another way to look at this integral which gives us a good opportunity to revisit Riemann sums. Since we know the integral diverges (work out the previous limit if you haven't already done so), we seek to approximate by a right-hand approximation, since if we can show that a sequence of right-hand approximations diverges, so must the integral. Don't forget how we proved the Integral Test!

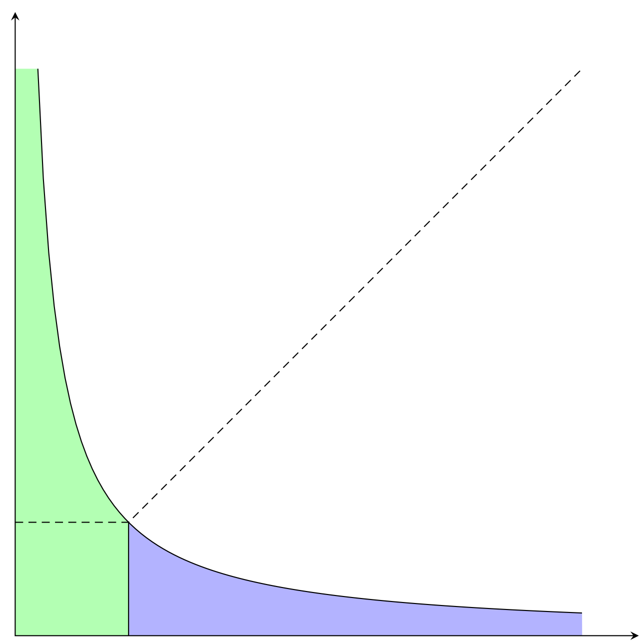
So let's divide the interval $[0, 1]$ into n equal subintervals. Then the kth subinterval is just

$$\left[\frac{k-1}{n}, \frac{k}{n}\right]$$

as we are taking a right-hand approximation. Now the width of each subinterval is clearly $1/n$, so we approximate (in fact, underestimate) the value of this integral by

$$\int_0^1 \frac{1}{x}\,dx > \sum_{k=0}^{n} \frac{1}{n} \cdot \frac{1}{k/n} = \sum_{k=0}^{n} \frac{1}{k}.$$

But wait! This is just a partial sum of the harmonic series, which we already know diverges. Hence the integral diverges as well. This should perhaps come as no surprise, however, since the function $y = 1/x$ is symmetric about the line $y = x$. Since we know that the area between the horizontal asymptote and the x-axis grows without bound, so must the area between the vertical asymptote and the y-axis. The integrals $\displaystyle\int_0^1 \frac{1}{x}\,dx$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ are compared in the drawing below, where the shape of the green region differs from that of the blue region by a unit square.



Of course such behavior is not limited to this one example.

||| Exercise 98: By considering symmetry about the line $y = x$, discuss the convergence of

$$\int_0^1 \frac{1}{x^k}\,dx \quad \text{and} \quad \int_1^\infty \frac{1}{x^k}\,dx$$

when $k > 0$. □

Did you try the previous exercise? The ideas contained in its solution are important, so we'll take a few moments and work it out in case you didn't get it.

Suppose $k > 0$ with $k \neq 1$. Let's evaluate

$$\int_0^1 \frac{dx}{x^k} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^k} = \lim_{a\to 0^+} \frac{1}{1-k}(1 - a^{1-k}).$$

It is clear that if $k < 1$, this limit exists and it equal to $1/(1-k)$, while if $k > 1$, the limit diverges (DNE($+\infty$)). An easy calculation shows that this integral also diverges when $k = 1$.

Now let's explore the symmetry of the situation. We'll look at a specific case for concreteness, but the geometry of the general case is identical. We will summarize the general case, however, at the end.

Consider
$$\int_1^\infty \frac{1}{x^2}\,dx.$$
Of course we know that this integral converges to 1. Now if we reflect the graph of $y = 1/x^2$ about the line $y = x$, we get the inverse function, $y = 1/\sqrt{x}$. Now $y = 1/x^2$ approaches the x-axis as x gets larger, so $y = 1/\sqrt{x}$ approaches the y-axis in a similar way as $x \to 0^+$, since the graphs are reflected across the line $y = x$ (that is, they are inverse functions). Draw a picture so you can see this!

Let's use this idea to evaluate
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx.$$
Using the same picture you drew before, note that the area under consideration is the same as the area underneath $y = 1/x^2$ on the interval $[1, \infty)$, *plus* the area of a unit square. Hence this integral must evaluate to 2, as we just showed that it did in a previous result with $k = 1/2$.

So, in general, we have that $y = 1/x^k$ and $y = 1/x^{1/k}$ are inverse functions. Moreover, when $k > 1$, we have
$$\int_0^1 \frac{1}{x^{1/k}}\,dx = \frac{1}{1-\dfrac{1}{k}} = \frac{k}{k-1} = 1 + \frac{1}{k-1},$$
so that
$$\int_0^1 \frac{1}{x^{1/k}}\,dx = 1 + \int_1^\infty \frac{1}{x^k}\,dx.$$

||| Exercise 99: Suppose that f is a decreasing function defined on $[1, \infty)$ such that
$$\int_1^\infty f(x)\,dx$$
exists. Then the x-axis must be a horizontal asymptote of f. Further, suppose that $f(1) = 1$, and let $g(x)$ be the inverse function of f. Note that the domain of g must be the range of f; that is, $(0, 1]$. Show that
$$\int_0^1 g(x)\,dx = 1 + \int_1^\infty f(x)\,dx.$$
□

||| Exercise 100:

1. Evaluate $\int_0^1 \ln x \, dx$.
2. Using the technique illustrated in the previous exercise, evaluate this integral in a different way.

□

||| Exercise 101: Determine if $\int_0^{\pi/2} \tan x \, dx$ converges. □

||| Exercise 102: Find $\int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx$. What about $\int_{-\infty}^{\infty} \frac{x}{1+x^2} \, dx$? □

||| Exercise 103: Find

$$\int_0^{\infty} (1 - \tanh x) \, dx.$$

□

||| Exercise 104: Consider the series

$$A = \sum_{n=1}^{\infty} 2^{-n}$$

with the notations given in Theorem 10.

1. Find A_{10}.
2. Find lower and upper bounds for A_{10} as suggested by Theorem 10. How good are these bounds?
3. Find an upper bound for the error R_{10}. Compare this to the exact error.

□

||| Exercise 105: Consider the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}.$$

Determine whether or not the series converges:

1. by comparing it to a series you already know, and

2. by using the Integral Test.

□

||| Exercise 106: Determine the convergence of the series

$$\sum_{m=1}^{\infty} \frac{1}{m(1+\ln m)^2}.$$

□

||| Exercise 107: Determine if the following series converges:

$$\sum_{n=1}^{\infty} \frac{\ln(n^2)}{\ln(2^n)}.$$

□

||| Exercise 108: Suppose that $p, q > 0$. In terms of p and q, give a criterion which indicates when

$$\int_0^1 \frac{dx}{x^p + x^q}$$

converges. □

11 Elementary Complexification

This is a good time to introduce you to the amazingly intriguing world of complex analysis. Truly mind-boggling, as you'll come to see. And the reason is this: nothing we've done so far prohibits the use of *complex* numbers. Taking limits (and hence finding derivatives), all the algebra, etc., is just as valid with complex numbers as with real numbers. It turns out that there is also a lot *more* going on than what we have just discussed...but I guess that's why you'll need to take a course in Complex Analysis.

Through the use of complex numbers, we find some surprising connections between the circular and hyperbolic trigonometric functions. For example, we see that

$$\begin{aligned}\sinh ix &= (ix) + \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} + \frac{(ix)^7}{7!} + \frac{(ix)^9}{9!} + \cdots \\ &= ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \frac{ix^9}{9!} - \cdots \\ &= i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\right) \\ &= i\sin x.\end{aligned}$$

Take a moment to think about this. We saw that the Maclaurin series for $\sin x$ and $\sinh x$ looked *very* similar, except for the occasional minus sign – in fact, another minus sign every four powers of x. And of course $i^4 = 1$ – so perhaps this is not so much a surprise.

Now we may write this last result as

$$\sin x = \frac{1}{i}\sinh ix = -i\sinh ix,$$

so that

$$\sin ix = -i\sinh i(ix) = -i\sinh(-x) = i\sinh x.$$

||| Exercise 109: Derive the corresponding results for $\cos x$ and $\cosh x$. □

The results of the previous exercise – that $\cosh ix = \cos x$ and $\cos ix = \cosh x$ – show that both $\cosh ix$ and $\cos ix$ are real! Perhaps this should not be so surprising. If $f(x) = x^2$, then also $f(ix) = -x^2$, which is also real. But don't jump to conclusions too quickly – $\cosh ix$ is a real number if x is *real.* If x is a complex number, then $\cosh ix$ may be either real or complex.

You might even be tempted to think that this behavior is a result of the fact that $\cosh x$ and $\cos x$ are even functions. And, well, you'd be wrong....

||| EXERCISE 110: Consider the following function, defined on the complex numbers (where a and b are assumed to be real numbers):

$$f(a+bi) = \begin{cases} a+bi, & a>0, \\ -a-bi, & a<0, \\ |b|i, & a=0. \end{cases}$$

Show that f is an even function. In addition, show that if x is real and $x \neq 0$, $f(ix)$ is *not* real. $\square$

The celebrated formula of Euler (many mathematical entities were named after him) is not far behind. We saw earlier that $e^x = \cosh x + \sinh x$, so that replacing x with $i\theta$ results in

$$e^{i\theta} = \cosh i\theta + \sinh i\theta = \cos\theta + i\sin\theta.$$

Of course DeMoivre's theorem is virtually immediate:

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \left(e^{i\theta}\right)^n \\ &= e^{in\theta} \\ &= \cos n\theta + i\sin n\theta.\end{aligned}$$

Of course note that this is valid for *any* n. Usually, though, this is written as a theorem for integers $n \geq 0$.

||| EXERCISE 111: Prove DeMoivre's theorem for integers $n \geq 0$ using mathematical induction. $\square$

One interesting consequence of this is the famous

$$e^{i\pi} = \cos\pi + i\sin\pi = -1,$$

so that for all x,

$$e^{x+2\pi i} = e^x e^{2\pi i} = e^x \cdot \left(e^{i\pi}\right)^2 = e^x \cdot (-1)^2 = e^x.$$

In other words, e^x is a periodic function, with period $2\pi i$! This doesn't seem possible, knowing e^x as intimately as we do, but it is certainly true in the complex world.

Now that we see how the circular trigonometric functions relate to the hyperbolic trigonometric functions via their Maclaurin series, the similarity between hyperbolic and circular trigonometric identities makes more sense. For example, we see that

$$\cosh^2 x - \sinh^2 x = \cos^2 ix - (-i\sin ix)^2 = \cos^2 ix + \sin^2 ix = 1.$$

Neat! And we also have

$$\sinh 2x = -i\sin(2ix) = -2i\sin ix\cos ix = -2i(i\sinh x)\cosh x = 2\sinh x\cosh x.$$

||| EXERCISE 112: Try deriving other formulas involving hyperbolic trigonometric functions by considering the analogous circular trigonometric formulas. □

||| EXERCISE 113: Suppose $f(x) = \dfrac{1}{1-x^2}$.

1. Write the Maclaurin series for $f(x)$ and $f(ix)$. Average them; that is, add the series together and divide by 2.
2. Find $\dfrac{1}{2}(f(x) + f(ix))$, and then find its Maclaurin series.

□

12 Taylor Series

We now return to our discussion of issues which arise while discussing intervals of convergence. Next, we tackle Taylor series.

Recall the graphical interpretation of the approximations $P_n(x)$: as n grew larger, the polynomials $P_n(x)$ tended to approximate $f(x)$ better and better near $x = 0$, and it also appeared that the approximations were better further and further out.

Think about these graphs and how well they approximate $f(x)$ near $x = 0$. Now shift all the graphs to the right c. The result is called a **Taylor series**. We simply have to work out the details.

||| EXERCISE 114: Find the Taylor series for e^x about the point $x = 1$. □

Did you get it? Instead of taking derivatives at $x = 0$, we take them at $x = 1$. And, instead of powers of x, we include powers of $x - 1$, since we are shifting our graphs 1 to the right. Note also that all derivatives of e^x at $x = 1$ are e, so we may factor that out of the resulting series. Thus, we have

$$\begin{aligned} e^x &= e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \cdots \\ &= e\sum_{n=0}^{\infty} \frac{1}{n!}(x-1)^n. \end{aligned}$$

So we have *another* series which converges to e^x. Well, let's be careful how we say that. If we actually multiply out all the powers of $(x - 1)^n$, we do in fact get the Maclaurin series for e^x. Let's see how this happens.

First, let's examine the constant term, which in the Maclaurin series is just 1. Now in the Taylor series for e^x about $x = 1$, we write the sum of the constants in each of the terms:

$$e - e + \frac{e}{2} - \frac{e}{3!} + \frac{e}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n e}{n!} = e\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e \cdot e^{-1} = 1.$$

As expected! Note that we had to use the Maclaurin series for e^x evaluated at $x = -1$ to replace the last sum with e^{-1}.

||| EXERCISE 115: Using the same method, find the coefficient of x in the series expansion of e^x about $x = 1$. □

||| EXERCISE 116: (May safely be omitted on first reading.) Let $k > 0$ be given. Using this method again, find the coefficient of x^k in the series expansion of e^x about $x = 1$. □

||| EXERCISE 117: Divide each side of the Taylor series for e^x about $x = 1$ by e. Explain your result. □

Later, we will use Taylor series without resorting to these verifications. In general, it's a lot of work! But it is important to see that there is no magic here; Taylor series are a way of representing the *same* function, just written in a *different* form.

Thus, we see that it is not difficult to write down the most general Taylor series for a function f expanded about $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This simply shifts the Maclaurin series by a. Modifying Theorem 4 for error estimates is likewise fairly straightforward:

Theorem 11: Let a function f be given which is defined on an interval I, and let $c \in I$ be given. For $n \geq 0$, let $P_n(x)$ be the nth order Taylor approximation to $f(x)$ centered at $x = a$. Then for all $x \in I$, if $K > 0$ is such that $\left|f^{(n+1)}(t)\right| \leq K$ for all $t \in I$, we have

$$|f(x) - P_n(x)| \leq \frac{K}{(n+1)!}|x-a|^{n+1}.$$

(Exact Form) Moreover, for all $x \in I$, there is a point $c \in I$, depending on x and a, such that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

So when might we want to use Taylor series? One very interesting application will be addressed in the next section. But for now, let's take a simple example: calculate $\sqrt{4.1}$ to three decimal places. Of course trying a Maclaurin series is not possible – all derivatives of $\sqrt{x}$ are undefined at $x = 0$! So we need to try a Taylor series expansion. Centered where? Keep in mind that we must evaluate potentially several derivatives of $\sqrt{x}$, so choosing $x = 4$ might be a good idea since 4 is near 4.1.

||| Exercise 118: Calculate a third-order Taylor expansion to $\sqrt{x}$ centered at $x = 4$. Graph both $\sqrt{x}$ and the expansion and comment on what you observe. □

You should have obtained

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 + O((x-4)^4).$$

Using $x = 4.1$, it should be evident that only two or three terms are needed to get within 0.001 of $\sqrt{4.1}$.

We can in fact estimate the error of the first-order approximation. To do so, we need the absolute value of the maximum of the second derivative on a suitable interval. It suffices to use $x = 4$, since

$$\left|-\frac{1}{4}x^{-3/2}\right| = \frac{1}{4}x^{-3/2}$$

is decreasing on its domain. Thus, we may use $K = 1/32$, so that the error is no greater than

$$\frac{1/32}{2!}(4.1-4)^2 = \frac{1}{6400}.$$

It is easy to calculate the first-order approximation as $81/40$. For comparison, we note that

$$\frac{81}{40} - \sqrt{4.1} \approx 0.000154327, \qquad \frac{1}{6400} = 0.00015625.$$

Note that the latter estimate is the same error you would get if you considered the approximation to $\sqrt{4.1}$ as an alternating series.

||| Exercise 119: Note: This exercise should be done without a calculator.

1. Find the second-degree Taylor polynomial for $\sqrt{1+2x}$ about $a = 4$. Write your answer in exact form.
2. What value of x would you use to approximate $\sqrt{11}$ using this polynomial?
3. Use this value to approximate $\sqrt{11}$, simplifying your answer as much as possible.
4. Approximate the error for your approximation, simplifying as much as possible.

□

||| Exercise 120: Note: A calculator may be used for this exercise.

1. Find the third-degree Taylor polynomial for the function $\sqrt{5+x^2}$ about $a = 2$.
2. Use this polynomial to approximate $\sqrt{14}$.
3. Calculate the Lagrange error for your approximation, expressing your answer as a fraction in lowest terms.
4. In the interval $[2, 2.5]$, how close would x have to be to 2 so that your Taylor polynomial approximates $\sqrt{5+x^2}$ to within 10^{-9}? Write allowable values for x in interval notation, using five decimal places.

□

||| Exercise 121: Find the second-order Taylor polynomial $P_2(x)$ for $f(x) = x \sin x$ about $x = \pi$. Graph $f(x)$ and $P_2(x)$ on the same graph. □

||| Exercise 122: Find the Taylor polynomial $P_3(x)$ for $f(x) = \sin(3x)$ about $a = \dfrac{\pi}{6}$. Use this to approximate $f(x)$ when $x = \dfrac{\pi}{4}$, and approximate the error in this case. □

||| Exercise 123: Write out the Taylor series for $\sqrt{x}$ about $x = 4$ as an infinite sum. This essentially amounts to finding a formula for the nth derivative of $\sqrt{x}$. The following calculation might give you a hint as to how to proceed:

$$1 \cdot 3 \cdot 5 \cdot 7 = \frac{8!}{2 \cdot 4 \cdot 6 \cdot 8} = \frac{8!}{2^4 \cdot 4!}.$$

For full marks, prove your formula for the nth derivative is correct by using mathematical induction. □

||| Exercise 124: Note that in Exercise 121, although $f(x)$ is an even function, $P_2(x)$ contained odd powers of x. This is because an even function is symmetric about the line $x = 0$, and the Taylor expansion was found centered at π.

Define a function to be **even about** a if $f(a+x) = f(a-x)$ for all $x \in \mathbb{R}$. Show that if f is even about a, then a Taylor series expansion about a contains no odd powers of $x - a$. □

||| Exercise 125: State and prove the analogous result for a function f which is **odd about** a. □

||| Exercise 126: Consider the parabola $f(x) = 2x^2 - 6x - 5$. Using the definition of being even about a, find a such that f is even about a. Then find the Taylor expansion of f about a. □

||| Exercise 127: Let $f(x) = x^3 - 3x^2 + x + 1$. "Complete the cubic;" that is, find a such that f is odd about a. Then find the Taylor series for f about a. □

||| Exercise 128: Consider the cubic $f(x) = px^3 + qx^2 + rx + s$, with $p \neq 0$ (otherwise, f is not a cubic). Show that there exists a such that f is odd about a if and only if $9pqr = 2q^3 + 27p^2 s$. □

13 First Approximations

We now know enough to begin the long-awaited discussion of the accuracy of approximations, such as Euler. We recall that Euler's method was equivalent to the left-hand approximation for evaluating integrals, and that we learned many ways to approximate integrals – including the right-hand, midpoint, trapezoidal, and Simpson approximations. Using Taylor series, we will be able to discuss these approximations in a more quantitative way. We'll look at the left-hand and midpoint approximations in some detail, and then let you work out a few others.

Of course our first question must be: how can Taylor series be applied to this problem? Suppose a function f is given – and for the purposes of our discussion, assume it is differentiable as many times as is needed. Given any value for a, recall that an antiderivative of f is given by

$$F(x) = \int_a^x f(t)\,dt.$$

Now think for a moment: all the approximations listed above are intended to approximate $F(x)$, where this accumulation function is centered at a. But isn't that just what we've been doing? Approximating a function near a point? So what we need to do is to find a Taylor series for $F(x)$ about a.

Never mind that F is defined by an antiderivative – we can certainly differentiate it, can't we? We know that $F'(x) = f(x)$, and a moment's thought reveals that for $n \geq 1$, we have $F^{(n)}(x) = f^{(n-1)}(x)$. (Make sure you understand why we needed the qualification $n \geq 1$.) Of course, we also have $F(a) = 0$. Thus, we may write the Taylor series of F about a as

$$F(x) = 0 + f(a)(x-a) + \frac{1}{2}f'(a)(x-a)^2 + \frac{1}{3!}f''(a)(x-a)^3 \cdots \tag{14}$$

$$= \sum_{n=1}^{\infty} \frac{f^{(n-1)}(a)}{n!}(x-a)^n. \tag{15}$$

Let's first consider the simplest approximation – the left-hand approximation. Recall that such an approximation could be written

$$\int_a^b f(t)\,dt \approx \sum_{m=0}^{n-1} hf(a+mh), \quad h = \frac{b-a}{n}.$$

Take a moment to think about what the summation means. (Also, by this time, I hope you are not disturbed by the fact that we're using n in a slightly different way than we did for the series expansion for F. There just aren't enough letters in the alphabet for each one to have a specific meaning across all contexts.)

Recall our experimental evidence that the left-hand approximation is $O(h)$ – meaning that if we wanted to get 10 times more accurate, we needed to make h 10 times smaller. This wasn't very good, though. If we could find an $O(h^3)$ approximation, say, we could get 1000 times more accurate by making h 10 times smaller (or equivalently, making n 10 times larger).

Our strategy will be as follows: we'll first investigate how good our approximation is on each *sub*interval. Then we'll "add" all those approximations together in the usual way: if each subinterval yields an $O(h^2)$ approximation, say, then the entire interval yields an $O(h)$ approximation. The errors in the approximations on each subinterval are cumulative, so adding them together lowers the order of the approximation by 1. Let's do a few examples right away to see this.

In using Taylor series, we think of $[a, x]$ as a subinterval. The closer x gets to a, the better the approximation – just as the smaller h is, the better the approximation. Now we want to approximate $F(x)$ using a left-hand approximation. But this is just given by

$$L(x) = f(a)(x - a),$$

since $f(a)$ is the height of the rectangle and $x - a$ is the width of the rectangle. Compare this with (14). $L(x)$ is just the first-order approximation, so its error must be of the second order, or $O((x - a)^2)$. (We use the more accurate "$x - a$" here, as there is no h to be seen. What is significant is the order of the approximation – that is, the power of $x - a$.) But this is the error on a subinterval, so the entire approximation must be $O(x - a)$.

This might seem to be a rather simple example – but in looking at other approximations, the situation is decidedly more complex. To see this, let's investigate the midpoint approximation. Thus, we approximate $F(x)$ by

$$M(x) = f\left(\frac{a + x}{2}\right)(x - a).$$

Of course this is just the area of the rectangle whose height is determined by f at the midpoint of $[a, x]$.

Now we proceed to find a Taylor series expansion for $M(x)$. It is not so simple as the expansion for the left-hand approximation, since $M(x)$ is not linear, nor a polynomial (unless, of course, f is, in which case $M(x)$ is its own Taylor series).

Clearly $M(a) = 0$. We need to evaluate $M^{(n)}(a)$ for $n \geq 1$. Using the product and chain rules, we find that

$$M'(x) = f\left(\frac{a + x}{2}\right) + \frac{1}{2}f'\left(\frac{a + x}{2}\right)(x - a),$$

so that

$$M'(a) = f(a).$$

||| EXERCISE 129: Find $M''(x)$, $M''(a)$, $M'''(x)$, and $M'''(a)$. □

You should have found that

$$M''(x) = f'\left(\frac{a+x}{2}\right) + \frac{1}{4}f''\left(\frac{a+x}{2}\right)(x-a), \quad M''(a) = f'(a)$$

and

$$M'''(x) = \frac{3}{4}f''\left(\frac{a+x}{2}\right) + \frac{1}{8}f'''\left(\frac{a+x}{2}\right)(x-a), \quad M'''(a) = \frac{3}{4}f''(a).$$

Thus, we may write

$$M(x) = f(a)(x-a) + \frac{1}{2}f'(a)(x-a)^2 + \frac{1}{3!}\cdot\frac{3}{4}f''(a)(x-a)^3 + O((x-a)^4),$$

where the "$(O(x-a)^4)$" term means that the error is fourth-order since our approximation is third-order.

Now compare this expression with (14). Notice that the first- and second-order terms are identical, but that we're off on the third-order terms. In fact, we can explicitly calculate that

$$F(x) - M(x) = \frac{1}{24}f''(a)(x-a)^3 + O((x-a)^4).$$

Note that it is usual not to carry constants around with the $O((x-a)^4)$ term (recall that we subtracted it), since all we are interested in is the order of the approximation. (If it turns out the the constants are unusually large compared to a typical value of h, say, it may be necessary to be more precise – but that is not the case here.) Thus, we may say that

$$F(x) - M(x) = O((x-a)^3),$$

so that on each subinterval, the midpoint approximation is $O((x-a)^3)$. This implies, then, that the midpoint approximation is in fact $O((x-a)^2)$.

Let's take a look at what we have accomplished here. We have – fairly rigorously – *proved* that Euler's method is $O(h)$, and that the midpoint approximation to an integral is $O(h^2)$. No more making tables and guessing.

One rather important assumption which needs to be stated regards f. If f is differentiable as many times as necessary on $[a, x]$, this of course implies that f is continuous on $[a, x]$, as are all its derivatives – and moreover, the derivatives are bounded as well. In other words, f is well-behaved – as well-behaved as necessary to be able to find a Taylor approximation. For the moment, we will leave it at that. Approximations can also be used to estimate improper integrals as well, but we will not go into that here.

Now that we've taken a look at a few basic examples, it's time for you to try some.

||| EXERCISE 130: Show that the right-hand approximation is $O(h)$. □

||| EXERCISE 131: Show that the trapezoidal approximation is $O(h^2)$. □

||| EXERCISE 132: Suppose that

$$g(x) = \int_a^x f(t)\,dt$$

and

$$h(x) = \int_a^{2x} g(t)\,dt.$$

Find the second order approximation to $h(x)$ about $x = \frac{a}{2}$. □

14 The Limit Comparison Test

We now extend our previous discussion of the Direct Comparison Test. Perhaps a short exercise is in order to aid our memory.

||| EXERCISE 133: Using the Direct Comparison Test, show that both of the following series converge:

$$\sum_{n=2}^{\infty} \frac{1}{2^n + 3}, \quad \sum_{n=2}^{\infty} \frac{1}{2^n - 3}.$$

□

We note that finding the appropriate multiple of a convergent series to compare to can be a little tricky. Suppose, for example, we wish to prove that

$$\sum_{n=0}^{\infty} \frac{1}{2^n - n^2 - 10}$$

converges. Of course some of the terms are negative – and we would also need to find the appropriate multiple A such that

$$\frac{1}{2^n - n^2 - 10} < \frac{A}{2^n}$$

for sufficiently large n. Now this is not impossible, but it can be a little tedious to demonstrate. (Try it! We can choose $A = 4$, with the above inequality being valid for $n \geq 6$.)

However, there is another way to compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - n^2 - 10}$ to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$. Can you think about it for a moment and discover it?

Another way to see that these series are close is to observe that

$$\lim_{n\to\infty} \frac{1/(2^n - n^2 - 10)}{1/2^n} = 1.$$

This should make some intuitive sense – that this limit exists means that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - n^2 - 10}$ is in some sense "essentially" convergent, since it approaches a convergent sequence in the limit. And of course it is not necessary that this limit be 1; recall that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} ka_n = kL$ for any real number k.

This gives us the following theorem, where again, we assume the sequences in question are positive. Of course this means that they are "eventually" positive – in that we may ignore the first few terms if necessary and apply the result to the positive terms.

Theorem 12: (**Limit Comparison Test**) Suppose a_n and b_n, are sequences satisfying $a_n > 0$ and $b_n > 0$ for $n \geq 1$. Also suppose that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Then if L is finite and strictly positive, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge, or else they both diverge.

We'll actually prove this, since it isn't that hard. But before that, we must take a moment to address the "finite and strictly positive" qualifier to L. Well, we'll let *you* do that. Note that there are two cases to consider because of the symmetry of a_n and b_n in the theorem. In other words, if $\lim_{n\to\infty}(a_n/b_n) = 0$, then $\lim_{n\to\infty}(b_n/a_n)$ DNE($+\infty$).

||| Exercise 134: Exhibit positive sequences a_n and b_n such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0,$$

and one of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges and one diverges. □

||| Exercise 135: Exhibit strictly positive sequences a_n and b_n such that

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

does not exist, but approaches $+\infty$ (that is, DNE($+\infty$)) and one of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges and one diverges. □

Before moving on to do some practice, let's prove the Limit Comparison Test. So assume that two strictly positive sequences a_n and b_n satisfy

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L,$$

where L is finite and strictly positive. Now if we go out far enough – say for $n \geq N$, for some positive number N – we must have

$$0 < \frac{a_n}{b_n} < L + 1.$$

Without going into details about the definition of convergence, this should make sense. If the ratio of a_n to b_n approaches L, at some point it must be less than $L+1$ and stay there (otherwise it would not converge to L). Thus, for $n \geq N$, we have

$$0 < a_n < (L+1)b_n.$$

This means that

$$0 < \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n < \sum_{n=1}^{N-1} a_n + (L+1)\sum_{n=N}^{\infty} b_n.$$

A moment's thought here should convince you that if in fact $\sum_{n=1}^{\infty} b_n$ converges, then so must $\sum_{n=1}^{\infty} a_n$ – just look at the inequalities and apply the Direct Comparison Test.

||| EXERCISE 136: Since $L \neq 0$, consider that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L}.$$

Use an argument similar to that given above to show that if $\sum_{n=1}^{\infty} a_n$ converges, then so must $\sum_{n=1}^{\infty} b_n$. Note that this completes the proof of the theorem. □

||| EXERCISE 137: Consider the sequence

$$a_n = \frac{1}{n + \sin n}, \quad n > 0.$$

1. What difficulty is encountered in using the Direct Comparison Test with $b_n = 1/n$ to determine the convergence of $\sum_{n=1}^{\infty} a_n$?
2. Overcome this difficulty by using the Direct Comparison Test with a suitable multiple of $b_n = 1/n$.
3. Now use the Limit Comparison Test to determine the convergence of $\sum_{n=1}^{\infty} a_n$.

□

||| EXERCISE 138: As a result of some of the examples in the text and the previous exercise, it is tempting to think that if the Direct Comparison Text isn't easy to use, the Limit Comparison Test will save the day. But consider the series determined by the sequence

$$a_n = \frac{1}{n \ln n}, \quad n > 1.$$

1. Show that the Direct Comparison Test doesn't help here, since there is no k which satisfies $\dfrac{1}{n\ln n} > \dfrac{1}{kn}$ for all $n > 1$.
2. Show that the Limit Comparison is of little use, either.
3. Use the Integral Test to determine the convergence of $\displaystyle\sum_{n=2}^{\infty} a_n$.

□

||| Exercise 139: Determine whether the following series converges or diverges:

$$\sum_{m=1}^{\infty} \frac{\sqrt{m} - m}{m^4 + 3m}.$$

□

||| Exercise 140: Let a positive integer $p > 0$ be given. Define

$$a_n = \frac{1}{n + (\ln n)^p}, \quad n > 0.$$

Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$. □

||| Exercise 141: Consider the alternating harmonic series. Use this example to create two sequences b_n and c_n with the property that

$$0 < |b_n| \le |c_n|,$$

that $\displaystyle\sum_{n=1}^{\infty} c_n$ converges, but $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges. Such examples show why the Direct Comparison Test and the Limit Comparison Test require both sequences in question to contain only positive terms. □

15 New Series From Old

Up until this point, we have been looking at finding Taylor series for particular functions, as well as examining various properties of these series. It is now time to look at the various ways that series (either Maclaurin or Taylor) may be combined.

We saw a simple example of adding Maclaurin series earlier in observing that $\cosh x + \sinh x = e^x$. But a subtlety was glossed over – we added

$$\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right).$$

But of course series cannot *always* be added this way, as we saw with the alternating harmonic series:

$$\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) + \left(-\frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \cdots\right) = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right).$$

Certainly the right-hand side is convergent, but neither series on the left-hand side is convergent. We must be *very* careful when working with conditionally convergent series.

For most of the problems we deal with, such difficulties will not arise since, as you recall, most of the important series we encounter – $\sin x$, e^x, etc. – are absolutely convergent. An occasional remark or exercise will pop up now and again to keep us on our toes.

Adding and subtracting series is done as you might expect – simply add or subtract the corresponding powers of x, reminiscent of "combining like terms" when multiplying polynomials. For example, take

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\,,$$

presuming all the while that $|x| < 1$. Recall that geometric series are absolutely convergent, by the way. Then

$$\frac{1}{1+x} + \frac{1}{1-x} = 2(1 + x^2 + x^4 + x^6 + \cdots).$$

But of course

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2},$$

and thus the sum does in fact give us the correct Maclaurin series.

There are times when the need to add two series is disguised in another form.

||| EXERCISE 142: Find the Maclaurin series for $\dfrac{x-2}{2x^2+x-1}$, and find the interval of convergence. $\square$

There were a few morals to the previous exercise, but an important one is the fact that when manipulating series, there is no definitive way to determine the interval of convergence for the resulting series. A trivial example illustrates what may happen:

$$\frac{1}{1-x} - \frac{1}{1-x} = 0.$$

Two series with interval of convergence $(-1, 1)$ are subtracted – and the result is a series convergent over $\mathbb{R}$! So be careful, and don't make any rash assumptions....

||| EXERCISE 143:

1. If $f(x) = \dfrac{1}{1-x^2}$, find $f^{(42)}(0)$.
2. If $f(x) = \dfrac{x}{1-x^2}$, find $f^{(42)}(0)$.
3. If $f(x) = \dfrac{x^2}{1-x^2}$, find $f^{(42)}(0)$.

$\square$

Hopefully you noticed something interesting going on in the first two parts of the previous exercise. Did you notice that multiplying the series for $\dfrac{1}{1-x^2}$ by x resulted in the series for $\dfrac{x}{1-x^2}$? This may seem unremarkable at this point in our discussion, but think for a moment about what it means. Suppose that the Maclaurin series for $g(x)$ is given as usual:

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

We note this and all subsequent computations may be done with Taylor series as well, but using Maclaurin series at this point makes the calculations more transparent.

Now put $h(x) = x\,g(x)$. Then evidently, the Maclaurin series for $h(x)$ is given by

$$h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n = x \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^{n+1}.$$

Comparing powers of x^n, it is apparent that

$$\frac{h^{(n)}(0)}{n!} = \frac{g^{(n-1)}(0)}{(n-1)!}, \quad n \geq 1$$

so that

$$h^{(n)}(0) = ng^{(n-1)}(0)$$

for $n \geq 1$. This of course is convenient, since differentiating $h(x)$ each time involves the use of the product rule. However, even this is really not so bad.

||| EXERCISE 144: Using the product rule as necessary, find a formula for $h^{(n)}(x)$ in terms of $g(x)$. Prove your formula is valid using mathematical induction. □

You should have obtained the result

$$h^{(n)}(x) = x\,g^{(n)}(x) + ng^{(n-1)}(x), \quad n \geq 1.$$

It is then clear than upon substituting $x = 0$ in this expression, we do in fact get

$$h^{(n)}(0) = ng^{(n-1)}(0).$$

So the result is not all that mysterious.

Now this raises the following question: what if we multiply $g(x)$ by something more complicated than x? Indeed, the general case – $h(x) = f(x)g(x)$ – is something of a challenge, although we will take it up later. But it is interesting to look at one more specific case.

||| EXERCISE 145: Suppose $g(x)$ is given, and put $h(x) = x^2 g(x)$. Using similar techniques to those employed above, show that

$$h'(0) = 0, \quad h^{(n)}(0) = n(n-1)g^{(n-2)}(0), \quad n \geq 2.$$

□

||| EXERCISE 146: Using the previous result, find the Maclaurin series for $x^2 \cosh x$. Compare this to the termwise product of x^2 and $\cosh x$. □

Of course there is no reason to stop our investigation with x^2. We can keep going on, but that conversation will come a little later. Instead, we'll take a look at multiplying two *series* together.

The idea isn't all that difficult to understand. Recall that a series can be approximated by a polynomial of arbitrary degree. So the product of two series can be approximated by a product of polynomials, and we certainly know how to multiply polynomials together – use the distributive property and combine like terms. And that's exactly what we'll do.

So consider two series, say $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Suppose the product is some series $\sum_{n=0}^{\infty} c_n$. Then, collecting like terms, we may write

$$\begin{aligned}\sum_{n=0}^{\infty} c_n &= \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots\right) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3 + \cdots\end{aligned}$$

The pattern is hopefully evident. To get $c_n x^n$, we need to look at all the ways $a_j x^j$ and $b_k x^k$ can multiply together to get $c_n x^n$, which amounts to finding all possible ways that $j+k=n$, which is a straightforward task.

But a word of caution: in multiplying the two series this way, we are flagrantly disregarding the order of the terms – we're just combining them in a convenient way. So we'll adopt the usual approach to *define* the product of two series as above, and then worry about questions of convergence later.

So let's make another box:

DEFINITION 13: Suppose series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are given. We define the product of the sequences,

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$$

by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad n \geq 0.$$

Thus, we may write

$$\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} x^n.$$

Let's start with an example. Of course we have just done some; but writing x^2 as a series with all coefficients 0 except for the x^2 term is rather unenlightening. So let's look at the product

$$\frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2}$$

as a product of series. Writing these so we may use our definition gives

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} x^n, \quad \frac{1}{1+x} = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Thus $a_n = 1$ and $b_n = (-1)^n$ for $n \geq 0$. Thus for the product,

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} (-1)^{n-k}.$$

So there are just $n+1$ terms of alternating 1's and -1's, beginning with $(-1)^n$. So if $n+1$ is even, they all cancel, meaning that $c_n = 0$ if n is odd. And if $n+1$ is odd, the only surviving term is the first (or last, if you prefer) 1 (recall that n is now even), so that $c_n = 1$ if n is even. But this should come as no surprise, as in fact

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots ,$$

since it is a sum of a geometric series with common ratio x^2.

||| Exercise 147: Use this technique to find

$$\frac{1}{(1-x)^2}$$

as the product of two series. □

You should have found that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Now you might think this is curious; what is familiar about terms like $(n+1)x^n$? Maybe it would be more helpful to rewrite the sum as

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}.$$

The right-hand side looks exactly like a sum of derivatives; in fact, it seems that

$$\begin{aligned}\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n\, x^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{d}{dx} x^n \\ &= \frac{d}{dx} \sum_{n=1}^{\infty} x^n \\ &= \frac{d}{dx} \left(x + x^2 + x^3 + x^4 + \cdots \right)\end{aligned}$$

Can you see what's happening? The right-hand side is the derivative of $1/(1-x)$ – never mind that the 1 is missing, since its derivative is 0 – so we can put it back, if we like. Thus,

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx} \left(1 + x + x^2 + x^3 + x^4 + \cdots \right) \\ &= \frac{d}{dx} \frac{1}{1-x}.\end{aligned}$$

Now you might think this is a lot of trouble just to calculate a derivative, and you'd be right. But the result is suggestive – we can do calculus with series, as it appears that differentiating a series term by term does in fact give the appropriate derivative.

You likely noticed this already – termwise differentiate $\sinh x$, e^x, or $\sin x$. You get what you'd expect. But if you were paying attention, you noticed that we switched the "d/dx" with the "$\sum$" above. Now we know that the *finite* sum of derivatives is the derivative of the *finite* sum, but what of infinite sums? We don't have time to go into it right now, so that's just one more reason to take an Advanced Calculus course later on in life....

And of course if you can differentiate a power series, why not antidifferentiate? Yes, this is essentially equivalent (as we know from the Fundamental Theorem of Calculus), except that we must always be aware of adding a constant when integrating, as appropriate.

I suppose you might wonder what use there is for this, but there are some neat things you can do. For example, you might be aware that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

No? This is a very well-known series which can be used to calculate π. However, the convergence is *very* slow, as you might guess. Recall that in an alternating series such as this, the error is approximated by the absolute value of the first omitted term – and the terms in this series do not approach 0 very rapidly. In fact, the first 100 terms add to approximately 3.13159 – and no, that's no typo. So in practice, no one takes this series seriously....

And notice – no factorials in the denominators. Just odd numbers. And when might you get *just* odd numbers on the denominator? Maybe by antidifferentiating? If you'd like, stop for a moment before reading further and try to figure it out on our own. Otherwise, read on.

So we're looking at antidifferentiating *even* powers of x in order to get *odd* numbers on the denominator. Perhaps something like

$$\int (1 - x^2 + x^4 - x^6 + \cdots)\, dx.$$

But you immediately recognize this as the integral of a geometric series, or

$$\int \frac{dx}{1 + x^2}.$$

This certainly looks familiar – it's just $\arctan x$. Putting it all together, we have

$$\begin{aligned} \arctan x &= \int \frac{dx}{1 + x^2} \\ &= \int (1 - x^2 + x^4 - x^6 + \cdots)\, dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \end{aligned}$$

||| Exercise 148: Find the interval of convergence for this series. □

Now it's easy. Since we've found that the series converges for $x = 1$, we substitute in and get

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

||| Exercise 149: Suppose that the interval of convergence for the series

$$\sum_{n=0}^{\infty} a_n x^n$$

is $(-r, r)$. Show that the series formed by taking the termwise derivative *also* has an interval of convergence of $(-r, r)$, with perhaps the endpoints added. □

Another interesting application of these ideas involves the renowned alternating harmonic series. We will look at an example of summing the following rearrangement of this series:

$$S = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \cdots$$

And *no* we don't use up the even reciprocals before the odd reciprocals. Take a moment to convince yourself that *every* even reciprocal *does* occur somewhere in this series.

How do you go about finding such a sum? There is more than one way, but we'll take a path which exploits what we've learned so far. Let's first group some pairs together:

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots + \frac{1}{14} - \frac{1}{16} + \cdots$$

Note that it is legal to group the first two terms of each triplet since we are not changing the order of any terms, so that the sum is still S. Also take a moment to see that this is a natural step to take, as each odd denominator is just half the immediately subsequent even denominator.

At this point, you should be able to recognize this series! If you can't, that's OK – we'll look at the trick later. But rather than spoil the surprise....

Now let's combine the terms in pairs – which again maintains the proper order. We get

$$S = \frac{1}{4} + \frac{1}{24} + \frac{1}{60} + \frac{1}{112} + \cdots$$

Notice that each denominator is a multiple of 4, so it makes sense to write

$$\begin{aligned} S &= \frac{1}{4}\left(1 + \frac{1}{6} + \frac{1}{15} + \frac{1}{28} + \cdots\right) \\ &= \frac{T}{4}, \end{aligned}$$

where we define T to be the relevant series for convenience.

Finally, we note that the denominators factor:

$$T = \frac{1}{1 \cdot 1} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 7} + \cdots + \frac{1}{n(2n-1)} + \cdots$$

Now we're in a position to try a few things! Give our previous work with telescoping series, it might be tempting to write

$$\frac{1}{n(2n-1)} = \frac{2}{2n-1} - \frac{1}{n},$$

but this puts us back, in some sense, where we started. Write out the first several terms of the series rewritten this way and see that this is true.

So this doesn't get us very far. However, when we found the Maclaurin series for $\arctan x$, we saw that antidifferentiating in some sense brought exponents of x to the denominator. So we might try some clever antidifferentiation and see where this gets us.

Let's begin – as we did with $\arctan x$ – with

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

(To make the ideas clearer visually, we will write out the first few terms rather than use summation notation.) Antidifferentiating, we get

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

Note that the constant of integration must be 0. In addition, the interval of convergence here is $[-1, 1)$. We "gained" an endpoint by antidifferentiating.

Take a look back at T. We're looking at denominators like $3 \cdot 5$, $4 \cdot 7$, etc. How might we obtain these? You might be interested to take a moment or two to try and figure it out on your own. Otherwise, read on and all will be revealed....

The idea is this: since we are (almost) doubling 3 and 4 to get 5 and 7, we need to (almost) double the exponents of x. But this can easily be done by substituting x^2 for x in the previous equation, so that

$$-\ln(1-x^2) = x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \frac{x^8}{4} + \cdots$$

A moment's thought shows that antidifferentiating doesn't quite give us what we want: $3 \cdot 7$, $4 \cdot 9$, etc. The other number on the denominator is too large by 2. But this doesn't really present much of a problem – we just divide these two powers of x out before we antidifferentiate. Thus,

$$-\frac{\ln(1-x^2)}{x^2} = 1 + \frac{x^2}{2} + \frac{x^4}{3} + \frac{x^6}{4} + \cdots,$$

so that

$$-\int \frac{\ln(1-x^2)}{x^2}\,dx = x + \frac{x^3}{2\cdot 3} + \frac{x^5}{3\cdot 5} + \frac{x^7}{4\cdot 7} + \cdots$$

Now things are looking up! Except for that potentially unpleasant integral there....

||| Exercise 150: Find

$$\int \frac{\ln(1-x^2)}{x^2}\,dx.$$

□

Thus, we have

$$\frac{\ln(1-x^2)}{x} - \ln\frac{1-x}{1+x} = x + \frac{x^3}{2\cdot 3} + \frac{x^5}{3\cdot 5} + \frac{x^7}{4\cdot 7} + \cdots$$

Note that no constant of integration is necessary here; as a result of the following exercise, we see that both sides of this equation are 0 as x approaches 0.

||| Exercise 151: Show that

$$\lim_{x\to 0}\frac{\ln(1-x^2)}{x} = 0.$$

□

||| Exercise 152: Find the interval of convergence for this series. □

||| Exercise 153: Let

$$f(x) = \frac{\ln(1-x^2)}{x} - \ln\frac{1-x}{1+x}.$$

Find

$$\lim_{x\to -1^+} f(x), \qquad \lim_{x\to +1^-} f(x).$$

□

Can it be that we are actually done? As a result of the previous exercise, you should have come up with $T = 2\ln 2$, so that

$$S = \frac{2\ln 2}{4} = \frac{\ln 2}{2} = \ln\sqrt{2}.$$

And now for the spoiler: you will remember, of course, the comment that you already recognized the sum

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots + \frac{1}{14} - \frac{1}{16} + \cdots$$

Simply write

$$S = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right).$$

||| Exercise 154: Using the technique developed above, find the sum of the alternating harmonic series, and thus show that $S = \frac{1}{2}\ln 2$, as expected. □

Now it's your turn!

||| Exercise 155: Sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

in two different ways. One way uses the method we just encountered, while the other is *considerably* easier. □

16 The Root Test

There is one more test for convergence we will consider, called the **root test**. To see why this might be useful, let's consider the series

$$\sum_{n=1}^{\infty} \frac{e^n x^n}{n^n}.$$

What is your intuition about this series? If you had to make an educated guess at this point, what would you guess?

Now think about what test you'd use to *prove* your result. An easy way to show convergence is to note than for $n \geq 3$, we have $n^n \geq 3^n$, so that

$$\left|\frac{e^n x^n}{n^n}\right| \leq \left|\frac{ex}{3}\right|^n,$$

so that the series is bounded above by a geometric series for suitable x, which clearly converges. The difficulty is that the interval of convergence cannot be determined using the Comparison Test – although we do know that it contains the interval $(-3/e, 3/e)$.

So let's see how the Ratio Test applies, since we can determine the interval of convergence this way. We first observe that

$$\lim_{n\to\infty}\left|\frac{e^{n+1}x^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{e^n x^n}\right| = \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\cdot\frac{e}{x}\right|.$$

Now what do we do about this limit? The usual thing to do is first consider

$$\lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n} = \lim_{n\to\infty}(n+1)\cdot\frac{(n+1)^n}{n^n} = \lim_{n\to\infty}(n+1)\left(1+\frac{1}{n}\right)^n.$$

You might recognize the second expression in the limit. If not, here's a chance for you to refresh your memory.

||| Exercise 156: Using L'Hôpital's rule as appropriate, evaluate

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

□

You should have found that this limit is e. Of course this implies that

$$\lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \frac{1}{e}.$$

Thus, we may evaluate the limit

$$\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\cdot\frac{e}{x}\right| = \lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\frac{n^n}{(n+1)^n}\cdot\frac{e}{x}\right| = 0\cdot\frac{1}{e}\cdot\frac{e}{x} = 0,$$

so that the original series converges for all values of x.

Whew! That was a little involved, although certainly within our ability. It turns out that there is a simpler way to see this, and it derives from – you likely guessed it – the geometric series. Again.

Now recall that the Ratio Test *also* derived from considering a geometric series. In that case, the idea was that the ratio between two successive terms of the series approached a nonzero limit. Let's take another look at the geometric series

$$\sum_{n=0}^{\infty} ar^n = a\sum_{n=0}^{\infty} r^n.$$

(You'll see in a minute why the a was factored out.) Now of course the ratios

$$\frac{r^{n+1}}{r^n}$$

have a limit, which is r. But there is *another* way to consider terms of a geometric series: consider

$$\lim_{n\to\infty}\sqrt[n]{|r|^n} = |r|.$$

This may seem obvious – but the Ratio Test was, in the same sense, also obvious. Applied to series, we see that if

$$\lim_{n\to\infty}\sqrt[n]{|a_n x^n|} = r,$$

then for large n, we have

$$|a_n x^n| \approx r^n,$$

so that the tails of the series behave like a geometric series – which we know to be convergent. And given this approximation, we also see that the series must be *absolutely* convergent. Thus, we have

THEOREM 14: (**Root Test**) Let a series

$$\sum_{n=0}^{\infty} a_n$$

be given, and consider

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

If this limit L exists, then the series converges if $L < 1$ and diverges if $L > 1$. No conclusion may be drawn if $L = 1$.

Moreover, if this limit DNE($+\infty$), then the series diverges. If the limit does not otherwise exist, no conclusion may be drawn.

Just a note: recall the example for the Ratio Test which shows that when the limit does not otherwise exist, no conclusion may be drawn. And in addition, recall the discussion preceding the Ratio Test: if the limit here is 1, we need to know just how quickly the terms a_n go to zero, and the Root Test doesn't give us that information.

We'll briefly sketch the main idea of the proof, as it is not too difficult. Again, the strategy is to compare the series with an appropriate geometric series. So suppose that a series $\sum_{n=0}^{\infty} a_n$ is given, and further suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = r.$$

Now if $r < 1$, put

$$R = \frac{1+r}{2} < 1.$$

This means that if we go far enough out in the series – say when $n \geq N$ – we have $\sqrt[n]{|a_n|} < R$ (since R is larger than r), so that $|a_n| < R^n$. This means that

$$\sum_{n=N}^{\infty} |a_n| < \sum_{n=N}^{\infty} R^n = \frac{R^N}{1-R}.$$

Thus, $\sum_{n=0}^{\infty} |a_n|$ converges since a tail converges.

A similar argument holds if $r > 1$; simply define R in the same way and show that the series diverges.

Now let's see how this applies to the series

$$\sum_{n=1}^{\infty} \frac{e^n x^n}{n^n}.$$

Surely you haven't forgotten it already! Applying the Root Test, we are looking at the limit

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{e^n x^n}{n^n}\right|} = \lim_{n\to\infty} \left|\frac{ex}{n}\right| = 0$$

for all values of x. Hence, we see that this series converges for all real x.

That was really easy! That was because given the nature of the series we were looking at, the Root Test made the calculation simple. As with much of mathematics, there is an artistry to determining whether a series converges or not, as well as finding its interval of convergence. Often more than one test will apply, and it is up to you to find the one which is most efficient and elegant to apply. And who knows, you might even come up with your own test for convergence. After all, *somebody* had to make these theorems up. They didn't just fall from the sky....

||| Exercise 157: Show that for any positive number, $\lim_{n\to\infty} \sqrt[n]{a} = 1$. □

||| Exercise 158: Recall the result we obtained using the Integral Test with a series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^k}, \quad k > 0.$$

Can you obtain the same conclusion using the Root Test? □

||| Exercise 159: In considering the convergence of a series like

$$\sum_{n=1}^{\infty} \frac{n!}{n^n},$$

we might think of using the Ratio Test when considering the numerator, but the Root Test when considering the denominator.

In taking limits, it is often useful to employ *Stirling's approximation:*

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

Use Stirling's approximation and the Root Test to determine whether the sequence given above converges or diverges. □

||| Exercise 160: Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}.$$

□

||| Exercise 161: Show that the Maclaurin series for $\sinh x$ converges for all x using the Root Test. □

17 Generating Functions

Naturally you recall the world's most famous sequence:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2. \tag{16}$$

The Fibonacci sequence, of course. Your job is to do a little arithmetic and show that

$$\sum_{n=0}^{\infty} \frac{F_n}{2^n} = 2, \quad \sum_{n=0}^{\infty} \frac{nF_n}{2^n} = 10, \quad \sum_{n=0}^{\infty} \frac{n^2 F_n}{2^n} = 94. \tag{17}$$

Of course the result may strike you as surprising – why integers, and why 2, 10, and 94? And moreover, just how are you going to go about it? You are welcome to give it a try before reading on if you're courageous enough. All you need are a few series....

We need a slightly different approach here. Most of the examples we have been working with so far have been ones where we were *given* a function, and asked to *find* an approximation in the form of a Maclaurin or Taylor series expansion. The reverse problem is a common one for combinatorialists to ask, as we will see: given a series expansion, is there a simple function whose series expansion *is* the given one?

Let's take a look at the Fibonacci sequence, for example. Define

$$g(x) := \sum_{n=0}^{\infty} F_n x^n.$$

The function $g(x)$ is called the *generating function* for F. This is because if we know g, we can find its Maclaurin expansion and thus determine F. And once we know g, we'll be able to use it to solve lots of intriguing problems, such as deriving (17).

Two questions come to mind. The first and most obvious is: What is g? And the second: What is the interval of convergence for g? For clearly $g(0) = 0$, but that doesn't give us much information.

The trick is rather simple: just employ the recurrence relation given in (16):

$$F_n - F_{n-1} - F_{n-2} = 0.$$

For the uninitiated, we'll work out this example in some detail.

Of course, it is tempting to write

$$g(x) = \sum_{n=0}^{\infty} F_n x^n = \sum_{n=0}^{\infty} (F_{n-1} + F_{n-2}) x^n,$$

but there are issues when $n = 0$ or $n = 1$ – what are F_{-1} and F_{-2}? So we begin by rewriting g slightly:

$$g(x) = F_0 + F_1 x + \sum_{n=2}^{\infty} F_n x^n = x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) x^n.$$

We work with the sums a little, resulting in

$$g(x) = x + \sum_{n=2}^{\infty} F_{n-1} x^n + \sum_{n=2}^{\infty} F_{n-2} x^n = x + x \sum_{n=2}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} F_{n-2} x^{n-2}.$$

(Note: We can break apart the infinite sums because when they converge, they are (almost always) absolutely convergent.) Now why would we factor out the x and x^2 from the last two sums, respectively? Thinking about what the indices in a sum mean, it should be clear that

$$\sum_{n=2}^{\infty} F_{n-1} x^{n-1} = \sum_{n=1}^{\infty} F_n x^n, \qquad \sum_{n=2}^{\infty} F_{n-2} x^{n-2} = \sum_{n=0}^{\infty} F_n x^n.$$

If in doubt, consider the substitution $m = n - 1$ in the first sum:

$$\sum_{n=2}^{\infty} F_{n-1} x^{n-1} = \sum_{m+1=2}^{\infty} F_m x^m = \sum_{m=1}^{\infty} F_m x^m,$$

and then rewrite using n instead of m. (In general, the upper limit of a sum must also be changed when such a substitution is used, but the upper limit of "∞" is not affected in this case.) Thus, we have

$$\begin{aligned} g(x) &= x + x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=0}^{\infty} F_n x^n \\ &= x + x(g(x) - F_0) + x^2 g(x) \\ &= x + x\, g(x) + x^2 g(x), \end{aligned}$$

from which it readily follows that

$$g(x) = \frac{x}{1 - x - x^2}.$$

To find the radius of convergence, we seek x such that when

$$\lim_{n\to\infty} \left| \frac{F_{n+1} x^{n+1}}{F_n x^n} \right| = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} |x| < 1.$$

Thus the important question is: does

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

exist? Of course, the answer is yes.

||| EXERCISE 162: Find this limit. □

Now we have the radius of convergence, for we must have $\tau|x| < 1$, or

$$x \in \left(-\frac{1}{\tau}, \frac{1}{\tau}\right).$$

Of course we must try the endpoints; as it turns out, the series diverges at both endpoints (as we will see later). We might suspect this at the right endpoint since $g(1/\tau)$ is not defined. But even though $g(-1/\tau)$ *is* defined, this is not enough to guaranteed that the series converges there.

||| EXERCISE 163: Evaluate $g(-1/\tau)$. □

||| EXERCISE 164: Now that you know the interval of convergence, evaluate

$$\sum_{n=0}^{\infty} \frac{F_n}{2^n}.$$

□

||| EXERCISE 165: Show that

$$\sum_{n=0}^{\infty} \frac{nF_n}{2^n} = 10, \quad \sum_{n=0}^{\infty} \frac{n^2F_n}{2^n} = 94.$$

□

So that wasn't so hard, was it? And we've only begun! We can *also* use generating functions to find an *exact* formula for the terms of the Fibonacci sequence. You'll get to try this later; we'll illustrate the technique with a somewhat simpler case.

Let's begin with the sequence described by

$$H_n = H_{n-1} + 6H_{n-2}, \quad H_0 = 0, \quad H_1 = 1.$$

The first few terms of this sequence are

$$0, 1, 1, 7, 13, 55, \ldots$$

so that the generating function is given by

$$h(x) = \sum_{n=0}^{\infty} H_n x^n = x + x^2 + 7x^3 + 13x^4 + 55x^5 + \cdots$$

We will see how to use the generating function $h(x)$ to find an exact formula for H_n. Let's start the same way we did for the Fibonacci sequence, with

$$\begin{aligned}
h(x) = \sum_{n=0}^{\infty} H_n x^n &= H_0 + H_1 x + \sum_{n=2}^{\infty} H_n x^n \\
&= x + \sum_{n=2}^{\infty} (H_{n-1} + 6H_{n-2}) x^n \\
&= x + x \sum_{n=1}^{\infty} H_n x^n + 6x^2 \sum_{n=0}^{\infty} H_n x^n \\
&= x + x(h(x) - H_0) + 6x^2 h(x) \\
&= x + x\, h(x) + 6x^2 h(x),
\end{aligned}$$

so that

$$h(x) = \frac{x}{1 - x - 6x^2}.$$

The trick here is to notice that the denominator factors nicely, so that it is easy to rewrite $h(x)$ using partial fractions:

$$h(x) = \frac{1}{5}\left(\frac{1}{1-3x} - \frac{1}{1+2x}\right).$$

But we know the Maclaurin series for the fractions on the right-hand side, since they are geometric series. Thus, freely rearranging terms as necessary since geometric series are absolutely convergent,

$$\begin{aligned}
h(x) &= \frac{1}{5}\left(\sum_{n=0}^{\infty} (3x)^n - \sum_{n=0}^{\infty} (-2x)^n\right) \\
&= \frac{1}{5}\left(\sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} (-2)^n x^n\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{5}\left(3^n - (-2)^n\right) x^n.
\end{aligned}$$

Of course, this sum is *also* $\displaystyle\sum_{n=0}^{\infty} H_n x^n$. As a result, we have

$$H_n = \frac{1}{5}\left(3^n - (-2)^n\right), \quad n \geq 0.$$

Neat!

||| Exercise 166: Find an explicit formula for the sequence H given by

$$H_n = 4H_{n-1} - 3H_{n-2}, \quad H_0 = 2, \quad H_1 = 3.$$

□

||| Exercise 167: Find an exact formula for the Fibonacci sequence in terms of τ. □

||| Exercise 168: With the results of the previous exercise, show that that

$$\sum_{n=0}^{\infty} F_n x^n$$

diverges when $x = \pm 1/\tau$. □

||| Exercise 169: Find an explicit formula for the sequence given by

$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \quad n \geq 3, \quad H_0 = H_1 = 0, \quad H_2 = 1.$$

□

18 Second Approximations

Recall that earlier, we looked at the accuracy of the left-hand, right-hand, midpoint, and trapezoidal approximations. To review these ideas, let's start with a problem.

||| EXERCISE 170: Show that Simpson's rule in an $O(h^4)$ approximation as follows. Define

$$S(x) = \frac{1}{6}\left(f(a) + 4f\left(\frac{x+a}{2}\right) + f(x)\right)(x-a).$$

Proceed to calculate $S'(x)$, and then show by induction that

$$S^{(n)}(x) = \frac{1}{6}\left[n2^{3-n}f^{(n-1)}\left(\frac{a+x}{2}\right) + nf^{(n-1)}(x) + \left(2^{2-n}f^{(n)}\left(\frac{a+x}{2}\right) + f^{(n)}(x)\right)(x-a)\right] \tag{18}$$

for $n \geq 2$. Use this to find a simple expression for $S^{(n)}(a)$ for $n \geq 2$, and then find a fifth-order approximation to

$$F(x) = \int_a^x f(t)\,dt.$$

□

Hopefully you will appreciate that as approximations get more and more accurate, it is more difficult to prove their accuracy. I hope you will appreciate that if you weren't given a formula for $S^{(n)}(x)$, you would have been taking derivatives for a while....

Here is an excellent place where we can put technology to use. As it turns out, *Mathematica* (or your favorite symbolic computation package) can help with much of the calculation. Let's see how we would do the previous calculation using *Mathematica*.

First, begin by typing in the function we're considering:

```
S = (f[a] + 4 f[(a + x)/2] + f[x]) (x - a)/6
```

Now you can simply type `D[S,x]` and out will pop $S'(x)$. Neat! And entering `D[S,x,2]` will give you $S''(x)$.

It even gets better. The suffix "`/.x->a`" will substitute a for x in any expression, so we may get $S'(a)$ by typing `D[S,x]/.x->a`.

And even more amazing, we can get them all at once by putting them in a table! Just enter

```
Table[D[S, x, n] /.  x -> a, n, 6]
```

and see what you get. Then imagine how much work it would have been to do it yourself. Of course we did several examples earlier, and then saw that we could use induction to look at Simpson's rule. But we're going to consider several more approximations, and having *Mathematica* around will be a huge help to us. The important point is this: we've done enough examples so that we know we *could* work out the more difficult approximations – if we wanted to. It's rather like multiplying two six-digit numbers together. We know we could do it (maybe even correctly the first time!), but....

So let's look at reverse engineering – creating designer approximations. In other words, could we conceivably *derive* Simpson's approximation rather than use it off the shelf? Where did the 1/6 and 4/6 coefficients come from? Let's investigate.

Suppose we wish to create an approximation of the form

$$A(x) = \left(pf(a) + qf\left(\frac{a+x}{2}\right) + rf(x)\right)(x-a).$$

How might we proceed? Quite simply, upon looking at (14), we would like

$$A^{(n)}(a) = f^{(n-1)}(a)$$

for as many consecutive values of n as possible. But this is not a difficult task, as we have seen that we can easily obtain any derivatives of $A(x)$ we need. So try this in *Mathematica*. You should find that

$$\begin{aligned}
A'(a) &= (p+q+r)f(a), \\
A''(a) &= (q+2r)f'(a), \\
A'''(a) &= \left(\frac{3}{4}q + 3r\right)f''(a), \\
A^{(4)}(a) &= \left(\frac{1}{2}q + 4r\right)f'''(a), \\
A^{(5)}(a) &= \left(\frac{5}{16}q + 5r\right)f^{(4)}(a).
\end{aligned}$$

The first of these equation requires $p+q+r=1$ – but that is simply a statement that the coefficients of f are *weighted averages* of the values of the points at which we are evaluating f. This makes very good sense.

We may solve the next two equations,

$$q + 2r = 1, \quad \frac{3}{4}q + 3r = 1,$$

simultaneously for q and r, yielding (as we expect),

$$q = \frac{2}{3}, \quad r = \frac{1}{6}.$$

A quick substitution also shows that

$$\frac{1}{2}q + 4r = 1,$$

but then

$$\frac{5}{16}q + 5r = \frac{25}{24},$$

again as expected. And finally, we must solve for p from the first equation, so that $p = 1/6$.

Note that in seeing when the first three derivatives of A are what we want, we get an extra matching derivative "for free." Also note that $p = r$. This makes some intuitive geometrical sense as follows. Imagine that you draw f with a black marker so it can bleed through your paper. Draw an approximation using the method of your choice. Now look at the same approximation from the *other* side of the paper. Shouldn't it be the same? But when you do this, the left endpoint becomes the right endpoint, and vice versa. Thus it makes sense that the weights of the left and right endpoints should be the same.

Now it's your turn to create a designer approximation! But we're going to try something a little different this time.

||| Exercise 171: Consider an approximation which divides a subinterval into equal thirds – thus requiring that the function be evaluated at four points. What should the weights of these function values be to give the best approximation to $F(x)$? (Do not assume symmetry of the weights here – derive it!) What is the order of the approximation? □

||| Exercise 172: Repeat the previous exercise, but assume both symmetry of the coefficients and the fact that the coefficients are weights (that is, they sum to 1). What observations can you make? □

||| Exercise 173: Repeat the previous exercise, but use four subintervals. Assume symmetry of the coefficients in this case. But before doing so, think about this. Do you think that with four equal subintervals, you'll just get Simpson's rule on each half-interval? Or will you get something different? Why or why not? □

Now that we have addressed equal subdivisions of an interval in some detail, it is time to look at unequal subdivisions and a method called **Gaussian quadrature.** (Please note that this topic is discussed in the text in a rather different way. You should be able to read it and understand it.)

Note that when we created three subintervals, we subdivided the interval $[a, x]$ as follows:

$$a, \quad \frac{2}{3}a + \frac{1}{3}x, \quad \frac{1}{3}a + \frac{2}{3}x, \quad x.$$

But it also possible to still symmetrically subdivide the interval, but in a way that the middle piece is a different size than the pieces at either end. In this case, we would use a weight $\omega \in (0, 1)$, which creates the points

$$a, \quad \omega a + (1-\omega)x, \quad (1-\omega)a + \omega x, \quad x,$$

if $\omega > 1/2$. Then, assuming symmetry, we would have an approximation which looked like

$$A(x) = \left(pf(a) + \left(\frac{1}{2} - p\right) f(\omega a + (1-\omega)x) + \left(\frac{1}{2} - p\right) f((1-\omega)a + \omega x) + qf(x)\right)(x-a).$$

Using *Mathematica,* we find that

$$\begin{aligned}
A'(a) &= f(a), \\
A''(a) &= f'(a), \\
A'''(a) &= \frac{3}{2}((2-4p)\omega^2 + (4p-2)\omega + 1)f''(a), \\
A^{(4)}(a) &= 2((3-6p)\omega^2 + (6p-3)\omega + 1)f'''(a), \\
A^{(5)}(a) &= \frac{5}{2}(2p - (2p-1)(\omega-1)^4 - (2p-1)\omega^4)f^{(4)}(a).
\end{aligned}$$

||| Exercise 174: Setting coefficients of the derivatives of f at a to 1 as appropriate, solve for p and ω. □

You should have found that

$$\omega = \frac{5 \pm \sqrt{5}}{10}, \quad p = \frac{1}{12}.$$

Take a moment to convince yourself that both values of ω actually yield the same approximation. We choose the larger value of ω, $(5+\sqrt{5})/10$, so that the points in the interval move from left to right in the usual way.

It turns out that this approximation is $O((x-a)^7)$ on the subinterval (you should check this in *Mathematica*), so that it is $O((x-a)^6)$ on any given interval. Isn't this simply amazing? It seems difficult to imagine that with two well-chosen points – in addition to the endpoints – on any given interval, it is possible to get an approximation that good. Of course the function in question must be evaluated at these points, which involve square roots, so there is a complication there. But still, shifting slightly from subdividing into equal thirds gains us two orders of accuracy in our approximation.

||| EXERCISE 175: Find what would be called GQ_4 on the interval $[-1, 1]$ in our text, p. 408. □

||| EXERCISE 176: Using the methods of this section, derive the approximations for GQ_2 as given in the text, p. 408. (In the text, it is assumed that the approximation in on the interval $[-1, 1]$. Using the interval $[a, x]$ should result in the same relative sizes of the subintervals, scaled to an interval of length $x - a$. You may assume the subintervals have the appropriate symmetry.) □

||| EXERCISE 177: Using the methods of this section, derive the approximations for GQ_3 as given in the text, p. 408. See the instructions for the previous exercise. □

19 More New Series From Old

Earlier we looked at multiplying series and taking termwise derivatives and antiderivatives of series. Of course that was just the beginning.... It's time to take what we did a step or two further.

We began that discussion by looking at the Maclaurin series for $xg(x)$ and $x^2g(x)$. It wasn't obvious that just adding 1 or 2 powers of x to each term magically made the corresponding derivatives come out right. But of course it wasn't magic at all, just a little induction.

Well, now we're in for a *lot* of induction. By this point, you should be getting quite good at it.... So what happens if, instead of looking at derivatives of $xg(x)$ and $x^2g(x)$, we look at the more general $f(x)g(x)$? Can we apply the same techniques in this case? Certainly!

||| Exercise 178: Let $h(x) = f(x)g(x)$. Look for a pattern in the derivatives of h, and write this pattern using summation notation. □

You should have obtained an expression like

$$h^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x)g^{(n-j)}(x). \tag{19}$$

Hopefully you noticed than in working out the product rule each time and combining like terms, you were in fact algebraically forming Pascal's triangle. Now for the tricky part. Prove this formula is valid! This requires a careful induction argument.

It turns out that the type of argument is typical of combinatorial arguments; that is, roughly speaking, arguments involving binomial coefficients and their relatives. Likely you've not seen one before, so let's become familiar with what such an argument looks like before you tackle one on your own.

The quintessential example of such an argument is the proof by induction of the binomial theorem. In other words, for $n \geq 0$ and real numbers x and y, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{20}$$

It will look like this proof is long – but that's because each step will be carefully explained so that you understand what's going on. This should enable you to write one on your own!

So let's being with the base case: $n = 0$. This is easy, since in fact $1 = 1$. No problem.

Now assume that (20) is valid for n. We need to show that this formula is valid for $n = 1$. To begin, we write

$$\begin{aligned}(x+y)^{n+1} &= (x+y)(x+y)^n \\ &= x(x+y)^n + y(x+y)^n \\ &= x\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} + y\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k},\end{aligned}$$

where we have simply used the distributive property and substituted from the induction hypothesis. Now bringing the x and y terms into the sum amounts to some index-shifting:

$$(x+y)^{n+1} = \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k}.$$

Now step back a moment and look where we're going:

$$(x+y)^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n+1-k}.$$

In order to be able to combine like terms, we need to re-index the first sum:

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} &= \sum_{k=1}^{n+1}\binom{n}{k-1}x^k y^{n-(k-1)} \\ &= \sum_{k=1}^{n+1}\binom{n}{k-1}x^k y^{n+1-k}.\end{aligned}$$

Note that "n" never changes, but k is replaced everywhere by $k-1$. This brings us to

$$(x+y)^{n+1} = \sum_{k=1}^{n+1}\binom{n}{k-1}x^k y^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k}.$$

Now notice that the powers of x and y are the same – so we can combine like terms – but that the sums are now over different index sets! That's not a hard fix; simply take out the extra terms so like powers can be combined. Then

$$\begin{aligned}(x+y)^{n+1} &= x^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}x^k y^{n+1-k} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n+1-k} + y^{n+1} \\ &= x^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]x^k y^{n+1-k} + y^{n+1}.\end{aligned}$$

We now use the usual combinatorial identity to combine the expression in square brackets:

$$(x+y)^{n+1} = x^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n+1-k} + y^{n+1}.$$

Now that like terms have been combined, we now bring the extra terms back into the sum, noting that

$$x^{n+1} = \binom{n+1}{n+1} x^{n+1} y^{n+1-(n+1)}, \quad y^{n+1} = \binom{n+1}{0} x^0 y^{n+1-0}.$$

Thus,

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}.$$

Now this is just what we wanted to prove! And while it seems that it took a lot of work, such proofs are routine in the wonderful world of combinatorics. OK, so now it's time to work on your own.

||| Exercise 179: Prove that (19) is valid for $n \geq 0$ using mathematical induction. Note: Do not expect the similarities with the recent proof to be *exact*. But expect to use similar ideas when working with sums. □

Now let's use (19) to revisit some earlier results.

||| Exercise 180: Suppose a function $g(x)$ and an integer $k > 0$ is given, and put $h(x) = x^k g(x)$. By explicitly calculating $h^{(n)}(x)$ using (19), show that the Maclaurin series for $h(x)$ is given by

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^{n+k}.$$

□

Now let's look at an intriguing use of (19). We may use this result to prove many identities involving functions which may be represented by power series. We'll first take a look at a classic:

$$\sinh 2x = 2 \sinh x \cosh x.$$

We need simply to multiply the Maclaurin series for $\sinh x$ and $\cosh x$ together, double it, and then check that it is in fact equal to the Maclaurin series for $\sinh 2x$. No problem!

So let $f(x) = \sinh x$ and $g(x) = \cosh x$ in (19), so that

$$h^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} \sinh^{(j)}(x) \cosh^{(n-j)}(x). \tag{21}$$

To find Maclaurin series, we need to evaluate this at 0, giving

$$h^{(n)}(0) = \sum_{j=0}^{n} \binom{n}{j} \sinh^{(j)}(0) \cosh^{(n-j)}(0). \tag{22}$$

This turns out not to be too difficult to evaluate if we consider even and odd cases of n separately. This strategy should suggest itself readily, incidentally, since we do know that $\sinh 2x$ is an odd function.

The first observation to make is that derivatives of $\sinh x$ and $\cosh x$ at 0 are either 0 or 1, depending on which derivative is taken. Now suppose that n is even, and let's look at what happens as j goes from 0 to n. If j is even, then $\sinh^{(j)}(0) = \sinh 0 = 0$, so each of those terms drop out. But if j is odd, then since n is even, $n - j$ is *also* odd, so that $\cosh^{(n-j)}(0) = \sinh(0) = 0$. So these terms drop out as well, and we see that $h^{(n)}(0) = 0$.

When n is odd, you should take a moment to verify on your own that only the terms when j is odd survive, so that

$$h^{(n)}(0) = \sum_{j=0,\ j \text{ odd}}^{n} \binom{n}{j}.$$

||| EXERCISE 181: Find a formula for $h^{(n)}(0)$, and prove that your formula is valid. □

So now we know that $h^{(n)}(0) = 2^{n-1}$. Let's see where this gets us. We now write the Maclaurin series for $\sinh x \cosh x$. Keep in mind that only odd powers of x are needed to write this series. Then

$$\begin{aligned}
\sinh x \cosh x &= \sum_{n=0,\ n \text{ odd}}^{\infty} \frac{h^{(n)}(0)x^n}{n!} \\
&= \sum_{n=0,\ n \text{ odd}}^{\infty} \frac{2^{n-1}x^n}{n!} \\
&= \sum_{n=0,\ n \text{ odd}}^{\infty} \frac{(2x)^n}{2 \cdot n!} \\
&= \frac{1}{2} \sum_{n=0,\ n \text{ odd}}^{\infty} \frac{(2x)^n}{n!}
\end{aligned}$$

Finally, we recognize the last sum as the Maclaurin series for $\sinh 2x$. So we are done! It may seem a rather roundabout way to prove a simple identity involving hyperbolic trigonometric functions, but it reveals the versatility of Maclaurin series. And plus, we get to learn some neat combinatorics along the way....

||| EXERCISE 182: Using similar ideas, prove that

$$\cosh 2x = 2\cosh^2 x - 1.$$

□

||| EXERCISE 183: Prove another identity involving circular or hyperbolic trigonometric functions using this method. □

20 Miscellaneous Fun

OK, you've seen a *lot* about Maclaurin and Taylor series. Did you learn much? What follows is a series of exercises – some routine, some not. Some more interesting than others. But how to approach these problems is for *you* to decide. Rest assured that if you need a tool that is not already in your mathematical toolbox, it will be given to you.

||| EXERCISE 184: Let $p(x)$ be an arbitrary polynomial. Prove that the Maclaurin series for $p(x)$ is in fact $p(x)$ itself. Explain why this makes sense. □

||| EXERCISE 185: Prove the binomial theorem by finding the Maclaurin series for $f(x) = (x+y)^n$. Note: Treat y as a constant in this exercise, since f is a function *only* of x. □