Recall that we define the derivative $f'(x)$ as that function which satisfies

$$f(x+h) \approx f(x) + h\,f'(x)$$

for small h; more precisely,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x).$$

Example 1:

To find the derivative of $f(x) = x^3$, we see that

$$\begin{aligned}(x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3 \\ &= x^3 + h(3x^2) + o(h^2) \\ &\approx x^3 + h(3x^2).\end{aligned}$$

Comparing with $f(x+h) \approx f(x) + hf'(x)$, we see that

$$\frac{d}{dx}x^3 = 3x^2.$$

Example 2:

To prove the product rule, we use a similar strategy. If f and g are differentiable at x:

$$\begin{aligned}f(x+h)g(x+h) &\approx (f(x)+hf'(x))(g(x)+hg'(x)) \\ &= f(x)g(x) + h(f'(x)g(x) + f(x)g'(x)) + h^2f'(x)g'(x) \\ &= f(x)g(x) + h(f'(x)g(x) + f(x)g'(x)) + o(h^2).\end{aligned}$$

The term linear in h gives us, as we expect, the correct formula.

Example 3:

Given the limit

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1,$$

we have for small h, $\sin(h) \approx h$. Thus, for h small, we have

$$\begin{aligned}\sin(x+h) &= \sin(x)\cos(h) + \cos(x)\sin(h) \\ &\approx \sin(x) + h\cos(x),\end{aligned}$$

and so we have

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Example 4:

We may also use this idea to differentiate functions like $\sec(x)$. Write

$$\begin{aligned}
\sec(x+h) &= \frac{1}{\cos(x+h)} \\
&= \frac{1}{\cos(x)\cos(h) - \sin(x)\sin(h)} \\
&\approx \frac{1}{\cos(x) - h\sin(x)} \\
&= \frac{1}{\cos(x)(1 - h\tan(x))} \\
&= \sec(x) \cdot \frac{1}{1 - h\tan(x)}.
\end{aligned}$$

How do we work with an h in the denominator, as with

$$\frac{1}{1 - h\tan(x)}?$$

We recall the formula for an infinite geometric series:

$$\frac{a}{1-r} = a + ar + ar^2 + \cdots ,$$

as long as $|r| < 1$.

But for x fixed and h small, $h\tan(x) < 1$, and so we can approximate (to first order)

$$\frac{1}{1 - h\tan(x)} \approx 1 + h\tan(x).$$

So we have

$$\begin{aligned}
\sec(x+h) &\approx \sec(x) \cdot \frac{1}{1 - h\tan(x)} \\
&\approx \sec(x)(1 + h\tan(x)) \\
&= \sec(x) + h \cdot \sec(x)\tan(x).
\end{aligned}$$

This gives us the derivative of $\sec(x)$, as expected.

Some helpful formulas:

$$\lim_{h\to 0}\frac{\sin h}{h}=1,\qquad \lim_{h\to 0}\cos h=1,\qquad \lim_{h\to 0}\frac{\tan h}{h}=1.$$

These may be rewritten so that if h is small,

$$\sin h\approx h,\qquad \cos h\approx 1,\qquad \tan h\approx h.$$

$$\sin(x+h)=\sin(x)\cos(h)+\cos(x)\sin(h),\qquad \cos(x+h)=\cos(x)\cos(h)-\sin(x)\sin(h).$$

$$\tan(x+h)=\frac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)},\qquad \cot(x+h)=\frac{\cot(x)\cot(h)-1}{\cot(x)+\cot(h)}.$$

Problems to practice using these ideas:

1. Find the derivative of $f(x)=3x^2-4x+2$.
2. Prove the chain rule.
3. Find the derivative of $f(x)=\cos(x)$.
4. Find the derivative of $f(x)=\csc(x)$.
5. Find the derivative of $f(x)=\tan(x)$.
6. Find the derivative of $f(x)=\cot(x)$.