Continuity with non-contrived functions

In BC1, we tend to use contrived examples drawn by hand in problems about continuity, or perhaps piecewise defined functions. Such examples really require no thought on the part of the student in terms of deciding *where* the function is discontinuous, although we ask them to discuss continuity at the points in question. Similar comments apply for rational functions; simply check where the denominator is 0.

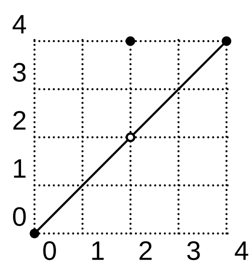
Yet using the floor function $y = \lfloor x \rfloor$, usually called the **greatest integer function,** allows for a discussion of continuity which occurs at a much deeper level.

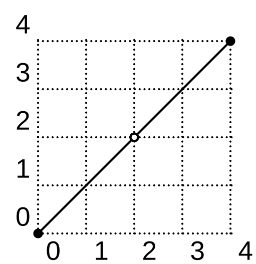
Before continuing, allow me to take a moment to stand on my soap box regarding the definition of continuity in OZ. We all know the three conditions. According to these conditions, since $\sqrt{x}$ is not defined at $x = -1$, then $\sqrt{x}$ is discontinuous at $x = -1$. Does this strike anyone else as absurd?

It is not difficult to propose that at a point x, a function f may be:

1. undefined;
2. defined and continuous;
3. defined and discontinuous.

Such a definition involves no concepts other than the ones already in play, and is the one most commonly used.





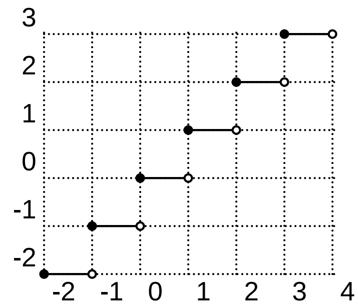
In the first example, we would say that there is a **jump discontinuity** at $x = 2$, while in the second, we would say that there is a **hole** at $x = 2$. Likewise, we would say that $y = 1/x$ has an **asymptote** at $x = 0$, while the function

$$y = \begin{cases} \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

has an **essential discontinuity** at $x = 0$.

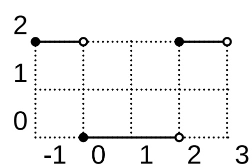
The function $y = 1/x$ **is** continuous on its domain. And in fact, I did tell my students that last semester. They were able to handle it. This definition also implies that sequences are continuous functions as well, but that conversation for another day.

Moving on. We introduce students to the function $y = \lfloor x \rfloor$, with jump discontinuities at the integers.



This allows for several questions concerning limits to be posed:

1. Find $\lim_{x \to -1^+} \lfloor x \rfloor$.
2. Find $\lim_{x \to -3^-} \lfloor -x \rfloor$. This question requires some thinking; students can be asked to sketch what's happening or write an explanation.
3. Where is $y = \lfloor 2x \rfloor$ discontinuous? This requires understanding that the floor function is discontinuous at the integers, and proceeding appropriately.
4. Where is $y = \lfloor x \rfloor^2$ discontinuous? $y = \lfloor x^2 \rfloor$? These are of course very different questions. There is a nice sublety here as well: $y = \lfloor x^2 \rfloor$ *is* continuous at $x = 0$, which might not be expected.
5. Where is $y = \lfloor x \rfloor^2 - \lfloor x \rfloor$ discontinuous? It is instructive to look at a graph of part of this function:

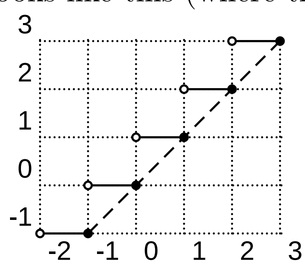
   

   Thus, we see that this function is in fact continuous at $x = 1$.
6. Graph $y = \sin \pi \lfloor x \rfloor$. Where is this function continuous? Of course this function is simply $y = 0$. Since
   $$\sin^2 \pi \lfloor x \rfloor + \cos^2 \pi \lfloor x \rfloor = 1,$$
   $\cos \pi \lfloor x \rfloor$ must take on the values -1 or 1; students can explore this by graphing $\cos \pi \lfloor x \rfloor$ as well. There are discontinuities here.

7. What is the domain of
$$y = \frac{1}{\lfloor x \rfloor^2 - 3\lfloor x \rfloor + 2}?$$
We must exclude $\lfloor x \rfloor = 1$ and $\lfloor x \rfloor = 2$, so that the domain consists of all real numbers except the interval $[1, 3)$.

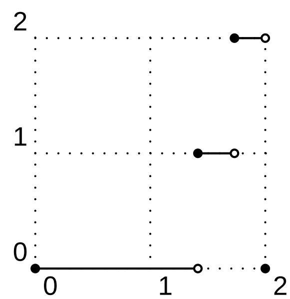
8. What is the domain of
$$y = \frac{1}{\lfloor x \rfloor^2 - 3x + 2}?$$
There is a nice subtlety here. y is undefined where $\lfloor x \rfloor^2 - 3x + 2 = 0$. But since $\lfloor x \rfloor^2 + 2$ is an integer, this is only possible when x is itself an integer, so that $\lfloor x \rfloor^2 - 3x + 2 = x^2 - 3x + 2 = 0$. Thus, y is undefined only when $x = 1$ and $x = 2$.

9. Graph $y = -\lfloor -x \rfloor$. It turns out this is the **ceiling** function $y = \lceil x \rceil$, the least integer greater than or equal to x. It looks like this (where the dashed line is $y = x$):



Of course this looks suspiciously like the floor function shifted to the left, except at the integers. In fact, it is not hard to show that
$$\lfloor x + 1 \rfloor - \lceil x \rceil = \begin{cases} 1, & x \in \mathbb{Z}, \\ 0, & x \notin \mathbb{Z}. \end{cases}$$
This is usually called the characteristic function for the integers. This can get students thinking; even *Mathematica* cannot get this one right....

10. Where is the function $y = \lfloor x^2 \rfloor - \lfloor x \rfloor^2$ discontinuous on the interval $[0, 2]$? Graph the function on $[0, 2]$.



The discontinuities occur at $\sqrt{2}$, $\sqrt{3}$, and 2.

It is hoped that these questions give an idea of the interesting ideas that can be addressed by considering the floor and ceiling functions. Their use requires thinking about discontinuity at a deeper level than what is usually asked for in BC1.