Calculus:
Understanding the Concepts

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Dr. Vince Matsko
Illinois Mathematics and Science Academy

Brief Description of IMSA

- Established in 1985;
- Funded by the State of Illinois;
- Grades 10–12;
- Residential school with 650 students, about half male, half female;
- Geared toward gifted students in mathematics and science;
- Strong programs in the humanities;
- Specific outreach to underserved students;
- Many summer programs for younger students;
- Professional development opportunities for teachers;
- Houses a Problem-Based Learning Network.

The typical IMSA student takes the following core sequences of courses:

- Precalculus: Mathematical Investigations II, III, and IV;
- Calculus: BC Calculus I, II and III.

Over one-third of the 650 IMSA students follow this path (data from 2010–2011):

Semester	Course			Totals
Fall	MI II	MI IV	BC II	
	84	99	68	251
Spring	MI III	BC I	BC III	
	86	79	68	233

Another 92 students placed in MI III or higher.

- Matrices: algebraic and geometric interpretations, the shoelace algorithm, operations on matrices, rotation and reflection matrices.
- Linear thinking: Rates of change, linear relationships, regression, arithmetic sequences.
- Functions: definitions, domain and range, composition and inverses, elementary transformations, odd and even functions.
- Exponential functions: Compound interest, radioactive decay and half-life, laws of exponents, geometric series.
- Combinatorics: Introduction to permutations and combinations, binomial theorem.

- Logarithms: definitions, graphs, rules of logarithms, solving equations, applications.
- Polynomials: graphs of monomials, roots, behavior of graphs at roots, factoring over real and complex numbers.
- Rational functions: domain and range, reciprocal graphs, asymptotes (vertical, horizontal, oblique, and polynomial), holes.
- Trigonometry: the unit circle, graphs of trigonometric functions, inverse trigonometric functions, solving equations.

- Sequences and series: arithmetic, geometric, and harmonic sequences and series; telescoping series; special series; applications to personal finance.
- Mathematical induction: introductory examples.
- Trigonometry: solving triangles, sine and cosine laws, deriving and using trigonometric formulas, proving identities, solving equations.
- Vectors: basic properties, solving word problems involving rates and forces.
- Polar coordinates: working with rectangular and polar coordinates and the corresponding representations of complex numbers, graphs of polar equations.

- Rates of change and Euler's method for obtaining displacement from velocity.
- Limits and the limit definition of a derivative.
- Limits and continuity.
- Graphs of functions and interpretations of first and second derivatives.
- Product, quotient, and chain rules.
- Implicit differentiation.
- Introduction to slope fields, differential equations, and Newton's law of cooling.
- Optimization.

- L'Hôpital's rule.
- The extreme value theorem and optimization; intermediate and mean value theorems.
- Newton's method.
- Parameterizations.
- Related rates.
- Area functions and The Fundamental Theorem of Calculus.
- Approximation using left-hand, right-hand, midpoint, and trapezoidal rules.
- Integration techniques: substitution, partial fractions, trigonometric substitution, trigonometric identities.
- Volumes of solids of revolution.

- Review of L'Hôpital's rule, sequences, and series.
- The direct comparison test for convergence.
- Maclaurin series and error analysis.
- The ratio test and intervals of convergence.
- Alternating series and their convergence.
- The integral test.
- Taylor series.
- The limit comparison test for convergence of series.
- The root test.
- Continued practice for the AP exam.
- Complex Maclaurin series.
- Generating functions.

- Limits and continuity: some warm-up exercises and examples.
- The derivative as a linear approximation, and its use in finding derivatives and formulas.
- Original Problems: getting students to think more conceptually.
- Lunch!
- Conceptual problems on traditional exams: examples and suggestions.
- Writing conceptual problems: drafts and critique.
- Different perspectives on Taylor series.
- Questions and answers.
- Sharing ideas and evaluations.

What are some ways to include a more conceptual approach to limits and continuity? We will

- Review important defintions.
- Take a quiz!
- Introduce the floor function.
- Work some examples.

Many textbooks use notations such as

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Students sometimes think the above limit "exists."

We write:

$$\lim_{x \to 0^+} \frac{1}{x} \qquad \text{DNE } (+\infty),$$

$$\lim_{x \to 0^-} \frac{1}{x} \qquad \text{DNE } (-\infty),$$

$$\lim_{x \to 0} \frac{1}{x} \qquad \text{DNE.}$$

A function f is said to be *continuous* at a if:

1. f is defined at a;

2. $\lim_{x \to a} f(x)$ exists;

3. $\lim_{x \to a} f(x) = f(a)$.

The function f is said to be *discontinuous* at a if f is defined at a but not continuous at a.

Careful! This is not the usual definition.

Why this definition?

- It is the usual definition in more advanced mathematics.

- If in fact $f(x) = 1/x$ is discontinuous at 0, then also $f(x) = \sqrt{x}$ must be discontinuous at $x = -1$.

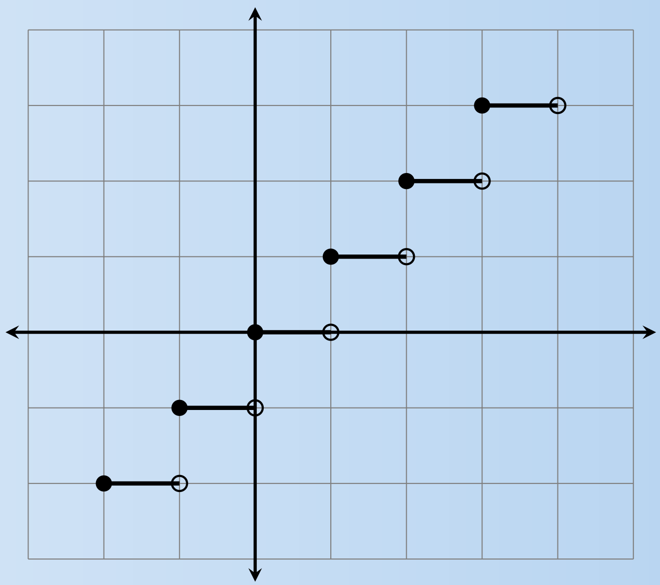
  It makes more sense to say – in each case – the function is undefined. Thus, $f(x) = 1/x$ is continuous on its domain.

- Thus, for a function f and a point a, exactly one of the following must hold: f is undefined at a, f is continuous at a, or f is discontinuous at a.

The *floor function* $f(x) = \lfloor x \rfloor$ is defined to be the greatest integer less than or equal to x. Thus,

$$\lfloor -10 \rfloor = -10, \qquad \lfloor 5.82 \rfloor = 5, \qquad \lfloor -\pi \rfloor = -4.$$

Below is part of the graph of the floor function:



This quiz was given to students as practice before an exam. They were given an extra credit quiz score for getting their question correct.

The quiz is available on my website.

Practice with the Floor Function

Now it's time for you to attempt some problems using the floor function.

The Derivative as Linear Approximation

Main concept:

$$f(x+h) \approx f(x) + hf'(x).$$

This is an equivalent form of the definition of a derivative; we may rewrite as

$$\frac{f(x+h) - f(x)}{h} \approx f'(x).$$

The symbol "$\approx$" may be made precise, if necessary.

We have

$$\begin{aligned}(x+h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3\\ &= x^3 + h(3x^2) + o(h^2)\\ &\approx x^3 + h(3x^2).\end{aligned}$$

The "$o(h^2)$" means that the expression is a function of h^2 (or some higher power of h). Since we will let $h \to 0$, such terms will become insignificant.

Comparing with $f(x+h) \approx f(x) + hf'(x)$, we see that

$$\frac{d}{dx}x^3 = 3x^2.$$

This allows for an easy proof of the product rule. If f and g are differentiable at x:

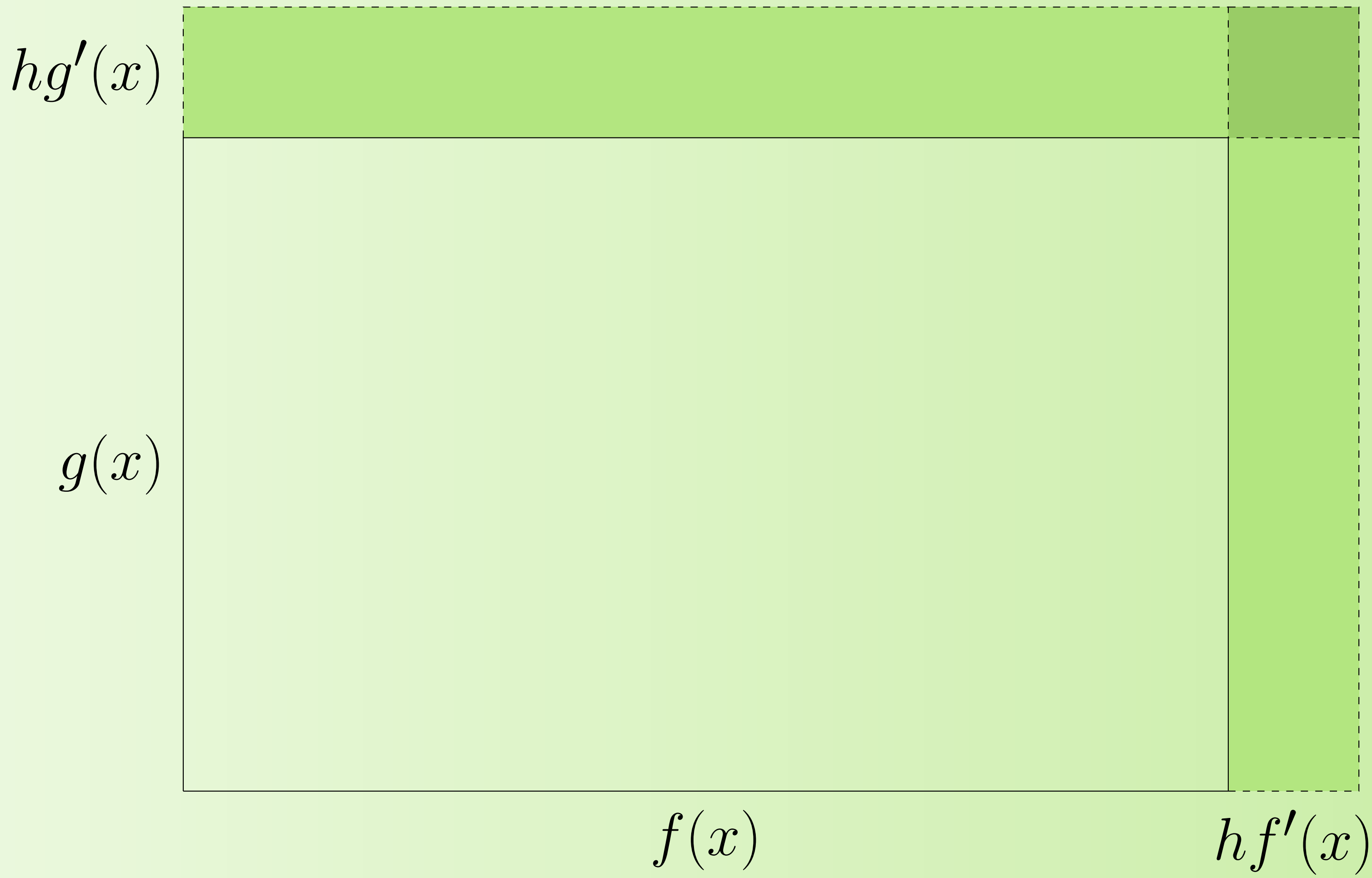
$$\begin{aligned} f(x+h)g(x+h) &\approx (f(x)+hf'(x))(g(x)+hg'(x)) \\ &= f(x)g(x)+h(f'(x)g(x)+f(x)g'(x)) \\ &\qquad\qquad\qquad\qquad + h^2 f'(x)g'(x) \\ &= f(x)g(x)+h(f'(x)g(x)+f(x)g'(x))+o(h^2). \end{aligned}$$

The term linear in h gives us, as we expect, the correct formula.

It is not necessary to be more formal at this level.

A Geometrical Illustration

$$f(x+h)g(x+h) - f(x)g(x) \approx hg'(x)f(x) + hf'(x)g(x)$$



A Note on the Quotient Rule

This technique may be applied to prove the quotient rule. But if $Q(x) = \dfrac{f(x)}{g(x)}$, we may write $f(x) = g(x)Q(x)$, so that

$$f'(x) = g'(x)Q(x) + g(x)Q'(x).$$

Solving for $Q'(x)$ gives

$$\begin{aligned} Q'(x) &= \frac{f'(x) - g'(x)Q(x)}{g(x)} \\ &= \frac{f'(x) - g'(x)\dfrac{f(x)}{g(x)}}{g(x)} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}, \end{aligned}$$

as expected.

Given the limit

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1,$$

we have for small h, $\sin(h) \approx h$. Thus, for h small, we have

$$\begin{aligned}\sin(x+h) &= \sin(x)\cos(h) + \cos(x)\sin(h)\\ &\approx \sin(x) + h\cos(x),\end{aligned}$$

and so we have

$$\frac{d}{dx}\sin(x) = \cos(x).$$

Note that this is equivalent to the limit definition, but is more transparent. This allows for more focus on concepts, less on details.

We may also use this idea to differentiate functions like $\sec(x)$. Write

$$\begin{aligned}\sec(x+h) &= \frac{1}{\cos(x+h)} \\ &= \frac{1}{\cos(x)\cos(h) - \sin(x)\sin(h)} \\ &\approx \frac{1}{\cos(x) - h\sin(x)} \\ &= \frac{1}{\cos(x)(1 - h\tan(x))} \\ &= \sec(x) \cdot \frac{1}{1 - h\tan(x)}.\end{aligned}$$

A Slight Diversion

How do we work with an h in the denominator, as with

$$\frac{1}{1 - h\tan(x)}?$$

We recall the formula for an infinite geometric series:

$$\frac{a}{1-r} = a + ar + ar^2 + \cdots\ ,$$

as long as $|r| < 1$.

But for x fixed and h small, $h\tan(x) < 1$, and so we can approximate (to first order)

$$\frac{1}{1 - h\tan(x)} \approx 1 + h\tan(x).$$

So we have

$$\begin{aligned}\sec(x+h) &\approx \sec(x)\cdot\frac{1}{1-h\tan(x)}\\ &\approx \sec(x)(1+h\tan(x))\\ &= \sec(x)+h\cdot\sec(x)\tan(x).\end{aligned}$$

This gives us the derivative of $\sec(x)$, as expected.

What is the underlying concept? That of *linear approximation.* Essentially, we are developing an intuition into when it is appropriate to approximate a function by its linear approximation.

A Final Observation

In more advanced calculus, this is necessary. For consider the following function which takes *vectors* as arguments:

$$f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}.$$

But we cannot take a difference quotient, since we cannot divide by vectors! But we can write

$$\begin{aligned} f(\mathbf{v}+\mathbf{h}) &= (\mathbf{v}+\mathbf{h}) \cdot (\mathbf{v}+\mathbf{h}) \\ &= \mathbf{v} \cdot \mathbf{v} + \mathbf{h} \cdot (\mathbf{2v}) + o(\mathbf{h})^2 \end{aligned}$$

Thus, we write

$$\nabla_{\mathbf{v}} f = 2\mathbf{v},$$

called the *gradient of* f. In advanced contexts, this is often the *only* way to find a derivative.

Practice with Linear Approximations

Now it's time for you to practice some problems using these techniques.

What is an Original Problem?

This assignment is a (possibly) short essay including the following components:

- Motivation
- (Conceptual) Problem Statement
- Problem Solution
- Reflection

A Gentle Grading System

To encourage students to be adventurous, they are told:

- Completing the assignment successfully with no errors will guarantee a grade of at least a B;
- Grades of C are assigned to clearly last-minute or careless efforts (usually only 10% of students earn a C);
- Many students earn an A$-$ for good conceptual problems;
- Fewer students earn an A, and there is the rare A$+$ for truly exceptional work.

Motivation: Stronger Student [BR]

I thought about the speed limit law conceptual problem from the last exam and I wanted to do something similar, but I didn't want the solver to just be able to use an equation to get the answer. I wanted them to have to think about it logically, and if they do that, the problem really isn't all that challenging.

Problem Statement: Stronger Student [BR]

$f'(x)$ is between 1 and 4 on the interval $[0,7)$ and between -2 and 0 on the interval $[7,11]$. If $f(0)=2$, what is the range of values of $f(x)$ at $x=11$?

Problem Statement: Stronger Student [MB]

You are on a planet far away, which has a different gravitational acceleration than Earth. The acceleration due to gravity on the far away planet was measured to be about 3.5 m/s^2. A hole needs to be made in the crust of the planet, and a scientist proposed the idea of shooting a rocket from the atmosphere to create the hole. The rocket's acceleration increases by $2\ m/s^2$. In order to properly penetrate the crust, the rocket needs to reach a velocity of $295\ m/s^2$. If the rocket started with an initial velocity of $0\ m/s^2$, how long does the rocket need to accelerate, and how far will the rocket travel in the time it takes it to accelerate to the proper velocity? Remember to account for the jerk (change of acceleration) in the rocket, and assume the acceleration of the planet aids the rocket in its trajectory towards the planet.

Motivation: Average Student [SR]

I am not that great at proofs nor I do I understand them very well, conceptually, when they are explained in graphical terms. Therefore, when we learned about the derivative of $\ln(x)$, I was very confused for a while. I wanted to try my hand at proving it algebraically and that's where this problem comes from.

Problem Solution: Average Student [SR]

PROBLEM: Prove that $\frac{d}{dx}\ln(x) = \frac{1}{x}$, using complete sentences.

This student researched the following approach:

$$\lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x+h}{x}\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right).$$

This was a different method that the usual one in textbooks, and one I had not seen before.

Problem Statement: Weaker Student [TY]

Create a function that is neither odd nor even and explain with proof why it isn't.

Reflection: Weaker Student [TY]

Describing how to do the minimum rate law problem was a lot harder than I expected. Especially, when I had to describe how to work it backwards. After explaining the problem, though, I am positive I have a much better understanding of how to solve these kinds of problems and use the two laws correctly.

As I worked on my first pair of original problems, I realized that problem writing only comes with a complete understanding of the topic in question and it can be used as a tool to measure your progress in the specific area. This is literally what I did towards the end of the semester; I first tried my best at understanding the topic, and checked whether or not I completely understood the topic or not by trying to write original problems.

I definitely felt that these problems were valuable. Apart from learning how to write effective problems, I felt that through these assignments, I learned how to think more conceptually.

After writing six of my own problems, I feel more prepared for the challenges that I will find in future math classes. Furthermore, I reviewed skills that I will need in the future, such as the logarithm rules and my algebra skills. Writing conceptual problems truly gave me a wholesome grasp of calculus.

A Few Remarks

- This is a good assignment for giving stronger students a challenge. They are able to create some really remarkable problems.

- For the average student, the gentle system of grading allows for them to be successful.

- For the weaker students, these problems give them an opportunity to review concepts they don't understand.

- The assignment can be adapted to many different classroom situations.

1. Using a graphical method, solve for a and b:
$$(1ab)_7 = (330)_a, \qquad (a42)_b = (31b)_8.$$
2. Describe and analyze the behavior of the graph of $y = \cos(\tan(x))$.
3. Find the period of $y = \sin^m(x)\cos^n(x)$ for positive integers m and n.
4. Describe graphs of the form $y = \sin(p\sin(qx))$.
5. Solve the equation $\sin(x) + 1/2 = 0$. (Note: this was done by shifting the unit circle up $1/2$ along the y-axis and considering the geometry of the resulting circle.)

6. Find the exact value of
$$\sum_{k=1}^{90}\cos^2(k)+\sum_{k=1}^{22}\tan^2(2k)-\sum_{k=1}^{44}\frac{1}{\tan^2(2k)}+\sum_{k=1}^{22}\frac{1}{\tan^2(2k)}.$$

7. Solve the quartic equation $x^4-8x^3+25x^2-46x+40=0$. (Note: This was done by finding a related cubic equation and using the method for solving cubics developed in the course Problem Sets.)

8. Suppose that a bishop and a knight are randomly placed on a chessboard. What is the probability that they attack each other?

Given the large class sizes typical in Thai schools and the many responsibilities of teachers, how can this type of assignment be introduced into the classroom? What training do teachers need in order to be able to give such assignments?

How can more conceptual problems be included on routine exams?

We begin by looking at an early exam in Calculus I.

Create a function f defined on all real numbers with the following properties:

- f is concave down on $(-\infty, 0)$;
- f is concave up on $(0, \infty)$;
- $f(x) \geq f'(x)$ for all real numbers x.

You do not need an explicit formula; a sketch is fine. Justify your answer appropriately.

Now it's your turn to try working through some more conceptual problems.

Practice Writing Conceptual Problems

Now it's your turn to try writing some conceptual problems.

Adventurous participants will present their problems for discussion.

A different approach to Taylor series has proved successful with both advanced and traditional calculus students.

A summary of significant differences from more traditional approaches follows.

The main weakness in most approaches is a lack of adequate motivation.

A typical L'Hôpital's rule problem:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(x)-x}{x^3} &= \lim_{x\to 0}\frac{\cos(x)-1}{3x^2}\\ &= \lim_{x\to 0}\frac{-\sin(x)}{6x}\\ &= \lim_{x\to 0}\frac{-\cos(x)}{6}\\ &= -\frac{1}{6}\end{aligned}$$

Why are there three application of L'Hôpital's rule?

We write

$$\sin(x) \approx x.$$

Antidifferentiating gives

$$-\cos(x) \approx \frac{1}{2}x^2 + C.$$

Since $\cos(0) = 1$, we have

$$\cos(x) \approx 1 - \frac{1}{2}x^2.$$

Repeating the process, we have

$$\sin(x) \approx x - \frac{1}{6}x^3.$$

Thus,

$$\sin(x) - x \approx -\frac{1}{6}x^3.$$

Thus, Maclaurin series are motivated by previous work.

Also, these approximations may be graphed to illustrate their usefulness in approximating either $\sin(x)$ or $\cos(x)$.

Polynomial Estimation Theorem: Let a function f be given which is defined on an interval I, and let $n > 0$ be given. Then for all $x \in I$, if $K > 0$ is such that

$$\left| f^{(n+1)}(t) \right| \leq K$$

for $t \in I$, we have

$$|f(x) - P_n(x)| \leq \frac{K}{(n+1)!} |x|^{n+1}.$$

What does this mean?

Recall:

$$|f(x) - P_n(x)| \leq \frac{K}{(n+1)!}|x|^{n+1}.$$

If K is constant, such as when $f(x) = \sin(x)$, then the Maclaurin series must converge for all x.

If K depends only on x, such as when $f(x) = e^x$, then the Maclaurin series must also converge for all x.

If K depends on n, such as with $f(x) = \sqrt{x}$, more work must be done to determine the interval of convergence.

Thus, this estimation formula is not only used, but is also discussed.

Comparison, ratio, root, limit comparison, integral, what do they all mean?

We use the convergence of essentially two different types of series:

- geometric series (ar^n);
- p-series (n^{-p}).

The ratio and root tests indicate that a series behaves *essentially geometrically.*

Euler considered the following “infinite polynomial” for $\sin(x)$:

$$\sin(x) = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

By comparing coefficients of the Maclaurin series for $\sin(x)$, we may deduce

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

By considering the series $\sin(ix)$, we see that

$$\sin(ix) = i\sinh(x).$$

By considering the series $e^{i\theta}$, we may deduce that

$$\begin{aligned} e^{i\theta} &= \cosh(i\theta) + \sinh(i\theta) \\ &= \cos(\theta) + i\sin(\theta). \end{aligned}$$

These and various other circular and hyperbolic trigonometric identities may be proved by considering Maclaurin series of a complex variable.

The Taylor series for $\int_a^x f(t)\,dt$ is given by

$$f(a)(x-a)+\frac{1}{2}f'(a)(x-a)^2+\frac{1}{3!}f''(a)(x-a)^3+\cdots .$$

By comparing this with the series for various approximations, such as the left-hand approximation:

$$f(a)(x-a),$$

or the midpoint approximation:

$$f\left(\frac{a+x}{2}\right)(x-a),$$

the order of the approximations may be calculated.

Let F_n represent the Fibonacci numbers. What is the sum

$$\sum_{n=0}^{\infty} \frac{F_n}{2^n}?$$

Or the sum

$$\sum_{n=0}^{\infty} \frac{nF_n}{2^n}?$$

By considering the generating function

$$f(x) = \sum_{n=0}^{\infty} F_n x^n$$

and finding both an explicit formula as well as the interval of convergence, such seemingly difficult questions can be easily answered.