The Greatest Integer (Floor) Function

Solutions to equations in one variable:

1. Since $\lfloor x^2 \rfloor$ and 2 are both integers, this means that x must also be an integer. Thus, $\lfloor x^2 \rfloor = x^2$, and so the equation may be rewritten as $x^2 - x - 2 = 0$. Thus, the solution set is $\{-1, 2\}$.

2. This problem is similar to the previous one. Since $\lfloor x \rfloor^2$ and 2 are integers, then x must also be an integer. Hence $\lfloor x \rfloor = x$, and we have the same solution set as before: $\{-1, 2\}$.

3. We must be careful in analyzing this solution. Since $\lfloor x \rfloor$ and 2 are integers, then so must be x^2. If x is an integer, we get the usual solutions: $x = -1$ and $x = 2$.

   But if x is *not* an integer, it must be of the form $\pm\sqrt{n}$, for some positive integer n, so that x^2 will be an integer. Note that if $n = 5$ so that $x = \pm\sqrt{5}$, we have

   $$5 - \left\lfloor \sqrt{5} \right\rfloor - 2 = 1 \quad \text{and} \quad 5 - \left\lfloor -\sqrt{5} \right\rfloor - 2 = 6.$$

   Increasing n only increases the "1" and the "6", so this means $n < 5$. It is an easy check to see that $n = 3$ yields the only possible additional solution: $x = \sqrt{3}$. Thus, the solution set is $\{-1, \sqrt{3}, 2\}$.

4. This is somewhat more complicated. Because of the $\lfloor x^2 \rfloor$ term, we examine intervals of the form $[\sqrt{n}, \sqrt{n+1})$ and $(-\sqrt{n+1}, -\sqrt{n}]$, since $\lfloor x^2 \rfloor$ is constant on such intervals. Realize again that n cannot be too large. A short table of calculations follows. Note that $\lfloor x^2 \rfloor - \lfloor x \rfloor - 2$ is 1 when $x \in (-\sqrt{2}, -1)$, but is 0 when $x = -1$; this accounts for the use of "or" in the table.

x interval	$\lfloor x^2 \rfloor - \lfloor x \rfloor - 2$	x interval	$\lfloor x^2 \rfloor - \lfloor x \rfloor - 2$
$(-\sqrt{2}, -1]$	1 or 0	$[\sqrt{2}, \sqrt{3})$	-1
$(-1, 0)$	-1	$[\sqrt{3}, 2)$	0
$[0, 1)$	-2	$[2, \sqrt{5})$	0
$[1, \sqrt{2})$	-2	$[\sqrt{5}, \sqrt{6})$	1

   This gives a rather unusual solution set: $\{-1\} \cup [\sqrt{3}, \sqrt{5})$. This problem illustrates the fact that simple polynomial equations can become fairly involved once the floor function is introduced.

5. We may factor as follows:

   $$\lfloor x \rfloor^2 - \lfloor x \rfloor - 2 = (\lfloor x \rfloor + 1)(\lfloor x \rfloor - 2) = 0,$$

   so that $\lfloor x \rfloor = -1$ or $\lfloor x \rfloor = 2$. This gives the solution set $[-1, 0) \cup [2, 3)$.