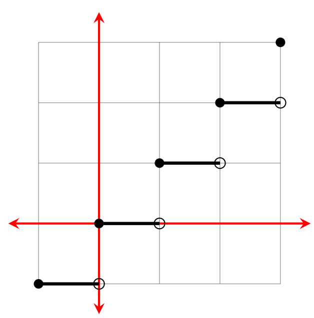
The Greatest Integer (Floor) Function

Define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. For example,

$$\lfloor \pi \rfloor = 3, \quad \lfloor 5 \rfloor = 5, \quad \lfloor -1.4 \rfloor = -2.$$

It is easy to see from the graph below why this is called a *step function.* We also call it the *floor function.*



We are used to solving the equation

$$ax = b, \quad a \neq 0$$

by finding that $x = b/a$. However, with the floor function, there are other possibilities.

For example, the equation $2\lfloor x \rfloor = 3$ has no solutions, since it is impossible for $\lfloor x \rfloor = 3/2$. Thus, we see that

$$a\lfloor x \rfloor = b, \quad a \neq 0$$

has solutions only when b/a is an integer. However, there are infinitely many solutions – the interval $[b/a, b/a + 1)$ contains all the solutions to this equation. For example, the equation $2\lfloor x \rfloor = 6$ has solutions in the interval $[3, 4)$.

But considering

$$\lfloor 2x \rfloor = 3$$

yields a different solution. Here, we must have

$$3 \leq 2x < 4,$$

so that the solutions are in the interval $[3/2, 2)$. Similarly, the solutions to the equation $\lfloor -2x \rfloor = 3$ satisfy the inequality $3 \leq -2x < 4$, and so lie in the interval $(-2, -3/2]$. So we see that solving linear equations involving the floor function requires more thought than solving usual linear equations.

Moving on to quadratic equations, the problems are even more interesting! Try to solve the following equations involving the floor function. You will see that even though the equations look similar, not all can be solved the same way. The problems are not necessarily listed in order of difficulty.

1. $\lfloor x^2 \rfloor - x - 2 = 0.$
2. $\lfloor x \rfloor^2 - x - 2 = 0.$
3. $x^2 - \lfloor x \rfloor - 2 = 0.$
4. $\lfloor x^2 \rfloor - \lfloor x \rfloor - 2 = 0.$
5. $\lfloor x \rfloor^2 - \lfloor x \rfloor - 2 = 0.$

If you found these problems interesting, try exploring graphs in the plane. For each equation, graph all points in the plane satisfying the equation.

1. $\lfloor x \rfloor + y = 1.$
2. $x + \lfloor y \rfloor = 1.$
3. $\lfloor x \rfloor + \lfloor y \rfloor = 1.$
4. $\lfloor x + y \rfloor = 1.$