Linear Algebra, Homework Quiz 4A 28 March 2018

1. Suppose $A = PDP^{-1}$, where

$$A = \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Find A^3 using this decomposition.

$$A^3 = PD^3P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^3 -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -8 & 16 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} -6 & 18 \\ 9 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -6 \\ -3 & 5 \end{bmatrix}$$

2. You are solving the recurrence $a_{n+2} = 10a_{n+1} - 24a_n$, $a_0 = 0$, $a_1 = 2$. You've already done a lot of work, and know that the relevant matrix has two eigenvalues: $\lambda = 4$ with eigenvector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\lambda = 6$ with eigenvector $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$. Now find a (nice!) formula for a_n.

$$P = \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 6 & -1 \\ -4 & 1 \end{bmatrix}$$

$$PD^nP^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 4^n & 0 \\ 0 & 6^n \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -4 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} 6\cdot 4^n & -4^n \\ -4\cdot 6^n & 6^n \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 6\cdot 4^n - 4\cdot 6^n & -4^n + 6^n \\ 24\cdot 4^n - 24\cdot 6^n & -4\cdot 4^n + 6\cdot 6^n \end{bmatrix}$$

Then: $$\frac{1}{2} \begin{bmatrix} 6\cdot 4^n - 4\cdot 6^n & 6^n - 4^n \\ 24\cdot 4^n - 24\cdot 6^n & 6^{n+1} - 4^{n+1} \end{bmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$= \begin{bmatrix} 6\cdot 4^n - 4\cdot 6^n & 6^n - 4^n \\ 24\cdot 4^n - 24\cdot 6^n & 6^{n+1} - 4^{n+1} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6^n - 4^n \\ 6^{n+1} - 4^{n+1} \end{pmatrix}$$

$$a_n = 6^n - 4^n.$$

3. If

$$\overset{U^{-1}}{\begin{bmatrix}1 & -5/2\\0 & 1\end{bmatrix}}\overset{D^{-1}}{\begin{bmatrix}1/2 & 0\\0 & 2/11\end{bmatrix}}\overset{L^{-1}}{\begin{bmatrix}1 & 0\\1/2 & 1\end{bmatrix}}A = I,$$

find an LDU decomposition for A. Also, Find A.

$$L = \begin{bmatrix}1 & 0\\-1/2 & 1\end{bmatrix} \quad D = \begin{bmatrix}2 & 0\\0 & 11/2\end{bmatrix} \quad U = \begin{bmatrix}1 & 5/2\\0 & 1\end{bmatrix}$$

$$A = LDU = \begin{bmatrix}1 & 0\\-1/2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 11/2\end{bmatrix}\begin{bmatrix}1 & 5/2\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\-1/2 & 1\end{bmatrix}\begin{bmatrix}2 & 5\\0 & 11/2\end{bmatrix}$$

$$= \begin{bmatrix}2 & 5\\-1 & 3\end{bmatrix}$$

4. Using an LDU decomposition, find $\det\begin{bmatrix}2 & 3\\-5 & 6\end{bmatrix}$. Do as *little* work as you need to!

$$2x + 3y = ? \qquad \times \tfrac{5}{2}$$
$$-5x + 6y = ?$$

$$2x + 3y = ?$$
$$0x + \tfrac{27}{2}x = ? \qquad \Rightarrow \quad D^{-1} = \begin{bmatrix}1/2 & 0\\0 & 2/27\end{bmatrix}$$

Then $D = \begin{bmatrix}2 & 0\\0 & 27/2\end{bmatrix}$, and $\det A = \det D = 2 \cdot \frac{27}{2}$

$= 27$

1. Suppose $A = PDP^{-1}$, where

$$A = \begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Find A^3 using this decomposition.

$$A^3 = PD^3P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^3 \frac{1}{3} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -8 & -16 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -6 & -18 \\ -9 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -6 \\ -3 & -5 \end{bmatrix}$$

2. You are solving the recurrence $a_{n+2} = 10a_{n+1} - 21a_n$, $a_0 = 0$, $a_1 = 4$. You've already done a lot of work, and know that the relevant matrix has two eigenvalues: $\lambda = 3$ with eigenvector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\lambda = 7$ with eigenvector $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$. Now find a (nice!) formula for a_n.

$$P = \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \quad P^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -1 \\ -3 & 1 \end{bmatrix}$$

$$PD^nP^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 7^n \end{bmatrix} \begin{bmatrix} 7 & -1 \\ -3 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 \cdot 3^n & -3^n \\ -3 \cdot 7^n & 7^n \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 7 \cdot 3^n - 3 \cdot 7^n & -3^n + 7^n \\ 21 \cdot 3^n - 21 \cdot 7^n & -3 \cdot 3^n + 7 \cdot 7^n \end{bmatrix}$$

Then: $$\frac{1}{4} \begin{bmatrix} 7 \cdot 3^n - 3 \cdot 7^n & 7^n - 3^n \\ 21 \cdot 3^n - 21 \cdot 7^n & 7^{n+1} - 3^{n+1} \end{bmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

$$= \begin{bmatrix} 7 \cdot 3^n - 3 \cdot 7^n & 7^n - 3^n \\ 21 \cdot 3^n - 21 \cdot 7^n & 7^{n+1} - 3^{n+1} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7^n - 3^n \\ 7^{n+1} - 3^{n+1} \end{pmatrix}$$

$$a_n = 7^n - 3^n$$

3. If

$$\overset{U^{-1}}{\begin{bmatrix} 1 & 5/3 \\ 0 & 1 \end{bmatrix}} \overset{D^{-1}}{\begin{bmatrix} 1/3 & 0 \\ 0 & 3/11 \end{bmatrix}} \overset{L^{-1}}{\begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix}} A = I,$$

find an LDU decomposition for A. Also, Find A.

$$L = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 11/3 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -5/3 \\ 0 & 1 \end{bmatrix}$$

$$A = LDU = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 11/3 \end{bmatrix}\begin{bmatrix} 1 & -5/3 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ 0 & 11/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$$

4. Using an LDU decomposition, find $\det \begin{bmatrix} 2 & 3 \\ 5 & -6 \end{bmatrix}$. Do as *little* work as you need to!

$$\begin{aligned} 2x + 3y &= ? \quad \times -\tfrac{5}{2} \\ 5x - 6y &= ? \end{aligned}$$

$$\begin{aligned} 2x + 3y &= ? \\ 0x - \tfrac{27}{2} y &= ? \end{aligned} \quad \Rightarrow \quad D^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & -2/27 \end{bmatrix}$$

Then $D = \begin{bmatrix} 2 & 0 \\ 0 & -27/2 \end{bmatrix}$, and $\det A = \det D = 2\left(-\frac{27}{2}\right)$

$$= -27$$