# Chapter 13

# Hypercubes in Four-Dimensional Space

What is the fourth dimension?

No, it's not time.

Well, maybe it is if you're studying physics. Even then, we have certain intuitive ideas about how time works. For example, if I imagine that it's 9:17 a. m. in San Francisco, then I know *what time it is* in any other city in the world. In New York City, it must be 12:17 p. m., since I add three hours to convert to Eastern Standard Time.

This assumption is just fine for getting along in daily life – and as far as most people are concerned, this way of thinking about time is *right.* But in fact, it works only because in our daily lives, we move around fairly slowly – at least compared to the speed of light. To understand what happens when particles do move close to the speed of light, you need to study *special relativity* – and here, our ordinary intuitions about time are no longer valid.

But we're interested in a fourth *spatial* dimension. How is this possible? Where is it? We are so used to living in a three-dimensional world, the idea of a fourth spatial dimension seems rather fantastic.

## 13.1 The Fourth Dimension

In 1884, Edwin Abbott's delightful novella *Flatland*[1] was published. The protagonist was none other than A Square, an inquisitive four-sided being living in a purely two-dimensional world called *Flatland.*

He was chosen as the Flatlander to receive a visit from a Sphere on the eve of the Third Millenium. This was an unnerving visit for A Square, since the Sphere kept suggesting he consider the direction upward, but there was no upward for A Square. There was North, South, East, and West, but A Square just couldn't fathom this direction, "upward."

Finally, the Sphere lifted A Square out of his two-dimensional world to show him the glory of Space. Quite a revelation!

We are in the same predicament as A Square when it comes to contemplating a fourth spatial dimension. Yes, we can look forward, backward, to our left and right, up and down, but nowhere else. It doesn't seem that there is enough room for a fourth dimension. Where would it be?

In a typical high school geometry class, you would likely have been introduced to points, lines, and planes – and perhaps were told that points are 0-dimensional, lines are 1-dimensional, planes are 2-dimensional, and that all objects of these types live in a 3-dimensional space. But while there were infinitely many points, lines, and planes, there was only *one* 3-dimensional space.

And while we do not encounter a fourth spatial dimension on a daily basis, we don't actually encounter points, lines, or planes, either. Can we actually see a point if it has no length? How could we possibly see a line if it has no width? It would be invisible. These geometrical ideas are in fact mathematical abstractions – and once we enter the world of mathematical abstraction, our universe becomes almost unimaginably vast. A. R. Forsyth wrote almost one hundred years ago:

> Mathematically, there is no impassable bar against adventure into spaces of more than three dimensions of experience.[2]

[1] Abbott, Edwin A. *Flatland: A Romance in Many Dimensions.* New York: Dover Thrift Edition. 1992.

[2] A. R. Forsyth, *Geometry of Four Dimensions,* Cambridge University Press, New York, 1930, p. vii.

To get a handle on how to imagine the fourth dimension, we'll look at one of the most popular and well-know denizens of that rarefied world – the hypercube, also known as a tesseract.

## 13.2 The Hypercube: Thinking By Analogy

How do we imagine a fourth spatial dimension? We have to think by analogy – go from zero dimensions (a point) to one dimension (a line segment), then from one dimension to two, from two to three, and then make the analogous leap from three dimensions to four. We'll do this in two different ways so you can see the process at work.

To create a line segment from a point, we think of it moving one unit (we need to be specific) along a "new" first dimension to create another point – these two points are vertices of the line segment, and one edge is created.

Now move this segment one unit along a second dimension which is *perpendicular* to the line segment.

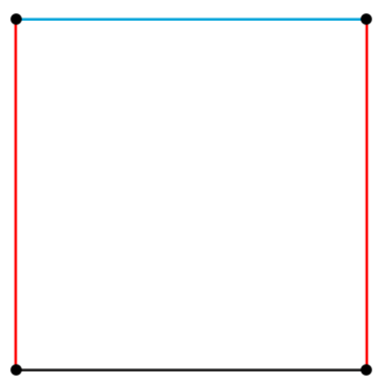

Figure 13.1: From line segment to square.

Now let's count. We have two vertices and one edge for each of the two segments, and each vertex creates another line segment (shown in red in Figure 13.1) as it moves from the bottom segment to the top segment. This gives a total of four vertices and four edges on a square – as expected.

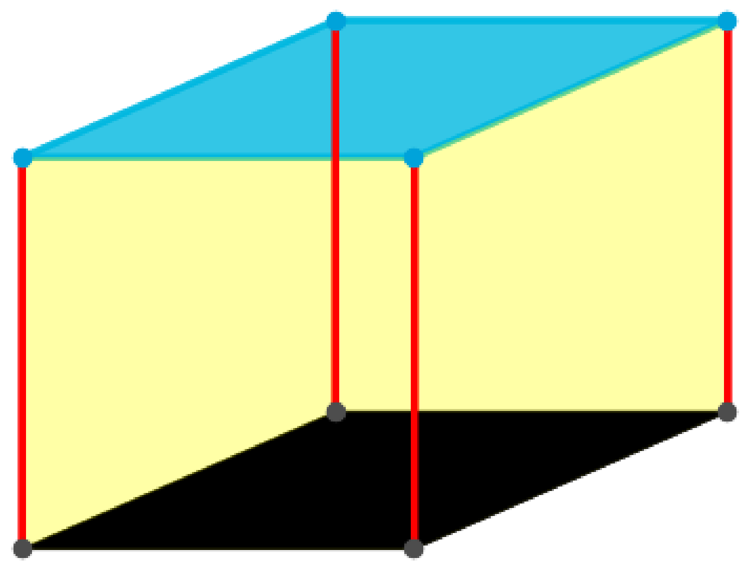

Figure 13.2: From square to cube.

Thinking by analogy, we now imagine a square moving *up* along a perpendicular dimension, as shown in Figure 13.2.

Counting vertices, edges, and faces, we have 8 vertices, 8 edges, and 2 faces from the bottom and top squares. Now each vertex of the bottom square creates an edge as it moves up (creating 4 more edges, shown in red), and each edge of the bottom square creates a new face (creating 4 new faces, shown in pale yellow). This gives a total of 8 vertices, 12 edges, and 6 faces. Notice the strategy: count the bottom and top figures, and then notice what is created by vertices and edges as they move along a perpendicular dimension.

Now it's time to extend this strategy into the fourth dimension. To do so, we need to imagine a cube moving *out* along a fourth spatial dimension – and of course, it is difficult to imagine because we are so used to our three-dimensional world.

Look at Figure 13.3. Let's think of the cube outlined in black as the base cube. Move this cube *out* along a fourth spatial dimension – so that each vertex creates an edge (shown in red) as it moves to the top cube (shown in

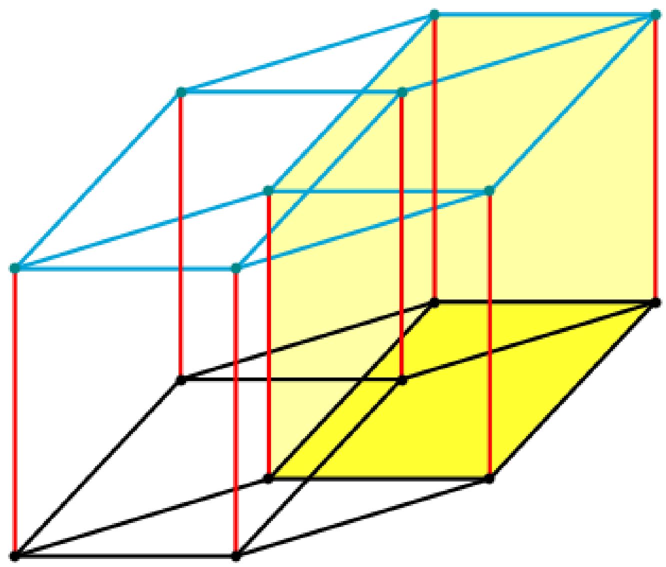

Figure 13.3: From cube to hypercube.

blue).

Before we start counting, we need to introduce a little terminology. The four-dimensional analogue of a polyhedron is called a **polytope,** and in addition to vertices, edges, and faces, we have three-dimensional **cells** on a polytope.

Now let's count the vertices, edges, faces, and cells on a hypercube in just the same way as in the previous two examples. For vertices, we have 16 total – 8 from the base cube, and 8 from the top cube. For the edges, we have 12 each for the base and top cubes, and each vertex of the base cube creates a new edge (shown in red). This gives a total of $12 + 12 + 8 = 32$ edges on the hypercube.

What about faces? Well, we have 6 faces each for the base and top cubes, and each edge of the base cube creates another square face (as it did when we created the cube). This gives us $6 + 6 + 12 = 24$ faces. Finally, to count the cells, we have the base and top cubes, and each square face of the base cube moves *out* to create another cube (such as the one shown in yellow, with the square from the base cube shown in darker yellow). This gives

$2 + 6 = 8$ cells (cubes) on the hypercube.

## 13.3 Another Analogy

But how do we know that the hypercube we imagined by analogy is actually *real*? It all depends upon what you mean by real.... We'll tackle that question in more depth later, but to help convince you even further, we'll arrive at the same counts of vertices, edges, faces, and cells on the hypercube by another method. First, we look at the square and cube in a different way, and think by analogy from there.

Let's start with the number of edges on a square this time. We'll start with one dimension less than the object we're examining – later we'll use the squares on a cube, and the cubes on a hypercube. Now to count vertices, we note that each edge has two vertices – for $4 \times 2 = 8$ vertices. But when we join the edges at vertices, two vertices merge, so we've overcounted by a factor of 2. Thus, there are $8/2 = 4$ vertices on a square (as we know).

What happens when we look at the cube in the same way? Let's start with the six squares. To count edges, we see that each square on the cube has 4 edges, for a total of $6 \times 4 = 24$ edges. But as with the vertices on the square, when we join two squares at their edges, two edges merge, so we've overcounted by a factor of 2. This means there are $24/2 = 12$ edges on the cube.

For vertices, we note that three squares meet at each vertex. So there are $6 \times 4 = 24$ vertices on the 6 squares – but when we join the squares together, *three* vertices merge to one. This means we've overcounted by a factor of 3, so there are $24/3 = 8$ vertices on a cube.

Now begin with the eight cubes on a hypercube. We note that the "8" comes from the sequence $2, 4, 6, 8, \ldots$ for 2 vertices on a segment, 4 edges on a square, 6 squares on a cube, etc. On the square, we saw that 2 vertices merged. On the cube, we observed that 2 edges merged, then 3 vertices. By analogy, on the hypercube, we should have 2 squares merging, 3 edges, and 4 vertices. This pattern continues into higher dimensions as well.

Let's use this analogy to check that we've counted correctly. First, we count squares. With 8 cubes, we have $8 \times 6 = 48$ squares. But since the cubes meet square-to-square, we've overcounted by a factor of 2, so that there are just $48/2 = 24$ squares on a hypercube.

Now on to the edges. Since 8 cubes contribute 12 edges each, there are 96 edges in total. But three cubes meet at each edge, so we've overcounted by a factor of 3. This implies that there are in fact $96/3 = 32$ edges on a hypercube.

Finally, we count the vertices. With 8 vertices on each of 8 cubes, we have 64 vertices all together. But four cubes meet at each vertex, so we have in fact overcounted by a factor of 4. This results in just $64/4 = 16$ vertices on a hypercube.

Thankfully, these are the same results as we obtained earlier. But how do we *know* they're correct? As we'll see in the next chapter, we can easily extend the idea of coordinates into four dimensions, and rigorously define a hypercube by its vertices. Then we'll see how each of the properties "proved" by analogy can in fact be proved using a four-dimensional coordinate system. In other words, we create a mathematical model for the hypercube, and derive its properties from that model.

## 13.4 Other Representations

As you might expect, there are other ways to represent a hypercube besides the object shown in Figure 13.3. We chose that particular representation because of the way we were thinking by analogy.

We'll look at two additional ways. The first is analogous to looking at a cube face on, as depicted in Figure 13.4.

Of course the inner and outer squares are the same size on a cube – but we can't help distorting faces of the cube when we make a two-dimensional sketch. Four of the squares are distorted into trapezoids here.

The analogous representation of the hypercube is shown in Figure 13.5. Here, the inner cube (in green with black edges) is in fact the same size as the outer cube (with blue edges); these cubes are directly opposite each other on the hypercube. This is directly analogous to the inner and outer squares in Figure 13.4.

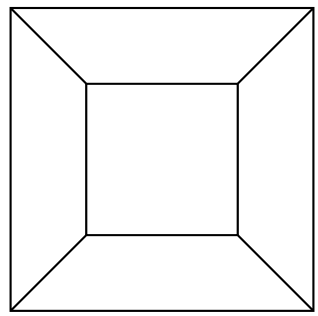

Figure 13.4: Viewing a cube face on.

Further, note how the other six cubes are distorted into frustums of square pyramids – you can easily see the trapezoidal faces, which we also saw in Figure 13.4. If you look carefully, you can count the 16 vertices and 32 edges. The 24 squares are a bit trickier – but begin with the 12 squares on the inner and outer cubes. Each of the other 12 squares contains exactly one black edge and exactly one blue edge – rather than the squares going up as in Figure 13.3, they are going out from the black edges to the blue edges. The perspective is different, but all the squares are there.

And of course there are the eight cubes – the inner cube, the outer cube, and the six frustums of square pyramids. All the elements of the hypercube are indeed present, but again, in a different perspective.

I did save the best for last – my favorite representation of a hypercube. It is shown in Figure 13.6.

I just love the symmetry of this image – the octagram inside the octagon. If you look at the left image in Figure 13.6, you'll see one of the eight cubes highlighted in blue. When each of the eight cubes is transparently colored in the right image, you'll see an interesting overlap of colors.

Now let's count the vertices, edges, faces, and cells in this figure. The 16 vertices are readily apparent in the inner octagram and the outer octagon. The 32 edges can be seen by counting eight edges from both the octagram and octagon, and two additional edges connecting each vertex of the octagon to two vertices of the octagram.

The squares are a bit trickier here as well – but eight are easily visible as undistorted squares sharing one edge of the octagon. But if you look

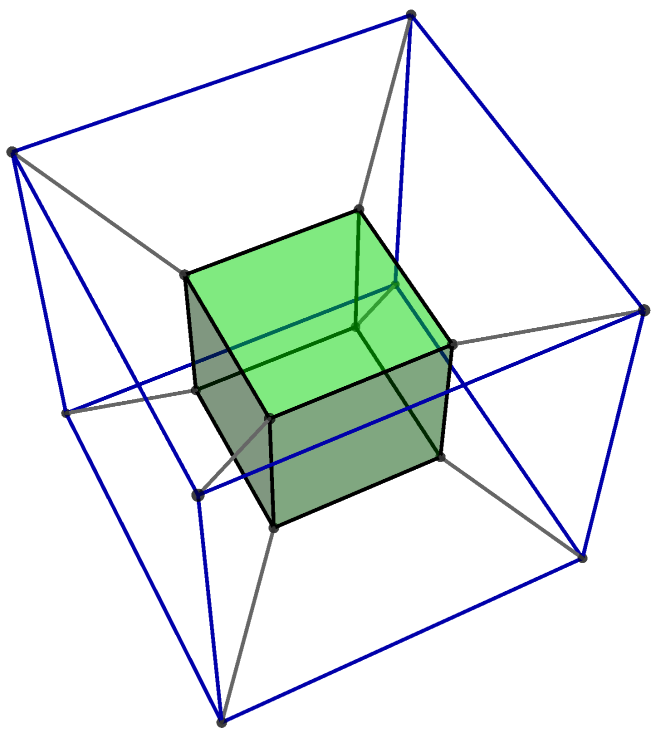

Figure 13.5: Another representation of a hypercube.

carefully, you'll also see 16 additional squares as rhombi in the figure. Eight of these rhombi each share two edges with the octagon, and the eight others each share two edges with the octagram. It might take staring for a minute, but they are all there.

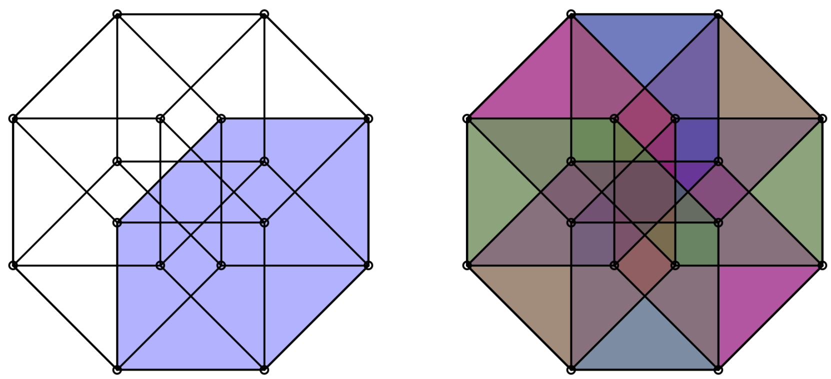

Figure 13.6: Yet another representation of a hypercube.

And finally, there are the eight cubes. As seen in the left image in Figure 13.6, three consecutive edges of the octagon are enough to determine one of the cubes. Since we can take three consecutive edges of an octagon in exactly eight ways, we have found the eight cells on the hypercube.

So again, all the elements of a hypercube are present – it just takes looking from the right perspective to see them all.

The next step? In the next chapter, we'll create a mathematical model for a hypercube, and then see how we can prove all the results we informally discussed in this chapter. The Forsyth quote bears repeating: "Mathematically, there is no impassable bar against adventure into spaces of more than three dimensions of experience."