1. Consider the sequence $2, 9, 16, 23, \ldots 7n+2$.

   (a) How many terms are there in the sequence? $n+1$

   (b) What is the second-to-last term? $7n-5$

   (c) Find the sum of all the terms in the sequence.

$$\begin{array}{cccc} 2 & 9 & \cdots & 7n+2 \\ 7n+2 & 7n-5 & & 2 \\ \hline 7n+4 & 7n+4 & \cdots & 7n+4 \end{array}$$

$$\frac{1}{2}(n+1)(7n+4)$$

2. Consider the linked system $A_{n+1} = 5A_n + B_n$, $B_{n+1} = 3A_n + 3B_n$, $A_0 = 1$, $B_0 = 3$. Find the quadratic characteristic equation you would need in order to solve this system. Then **STOP**.

$$\begin{aligned} A_{n+2} &= 5A_{n+1} + B_{n+1} \\ &= 5A_{n+1} + 3A_n + 3B_n \\ &= 5A_{n+1} + 3A_n + 3(A_{n+1} - 5A_n) \\ &= 8A_{n+1} - 12A_n \end{aligned}$$

Characteristic equation: $r^2 - 8r + 12 = 0$

3. Use polynomial fitting to find the nth term of the sequence $(a_n)_{n\geq 0}$

$$5, 10, 17, 26, 37, \ldots$$

5 7 9 11

2 2 2 ← Quadratic

$a_n = c_1 n^2 + c_2 n + c_3$

$a_0 = 5 = c_3$

$a_1 = 10 = c_1 + c_2 + c_3$

$a_2 = 17 = 4c_1 + 2c_2 + c_3$

$5 = c_1 + c_2$

$12 = 4c_1 + 2c_2$ $\cdot -2$

$-10 = -2c_1 - 2c_2$

$2 = 2c_1$

$1 = c_1 \rightarrow c_1 + c_2 + c_3 = 10$

$c_2 = 10 - 1 - 5 = 4$

$a_n = n^2 + 4n + 5$

4. You are given that the solution to the recurrence $g_{n+2} = 7g_{n+1} - 10g_n$, $g_0 = 0$, $g_1 = 1$, is of the form $c_1 \cdot 5^n + c_2 \cdot 2^n$. Find c_1 and c_2.

$g_0 = 0 = c_1 + c_2$

$g_1 = 1 = 5c_1 + 2c_2$ $\cdot -2$

$0 = -2c_1 - 2c_2$

$1 = 3c_1$

$c_1 = \frac{1}{3}$ $c_2 = -\frac{1}{3}$

5. Choose one of the following induction problems to solve.

   (a) Show that $\sum_{k=0}^{n} 3^k = \frac{1}{2}\left(3^{n+1} - 1\right)$ for all $n \in \mathbb{N}$.

   (b) Suppose the sequence L_n is given by the recurrence $L_{n+2} = L_{n+1} + L_n$, $L_1 = 1$, $L_2 = 3$. These are called *Lucas numbers*. Show that for all $n \geq 1$,

$$\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2.$$

(a) To see the base case, observe that $\sum_{k=0}^{0} 3^k = 1 = \frac{1}{2}(3-1)$.
Now assume the statement is valid for n. Then

$$\begin{aligned}
\sum_{k=0}^{n+1} 3^k &= \sum_{k=0}^{n} 3^k + 3^{n+1} \\
&= \frac{1}{2}(3^{n+1} - 1) + 3^{n+1} \quad \text{by assumption} \\
&= \frac{3}{2} \cdot 3^{n+1} - \frac{1}{2} = \frac{1}{2}(3^{n+2} - 1)
\end{aligned}$$

Thus, the statement is valid for $n+1$, and by induction, for all $n \in \mathbb{N}$.

(b) To see the base case, observe that $\sum_{k=1}^{1} L_k^2 = L_1^2 = 1 = 1 \cdot 3 - 2$.
Now assume the statement is valid for n. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} L_k^2 &= \sum_{k=1}^{n} L_k^2 + L_{n+1}^2 \\
&= L_n L_{n+1} - 2 + L_{n+1}^2 \quad \text{by assumption} \\
&= L_{n+1}(L_n + L_{n+1}) - 2 \\
&= L_{n+1} L_{n+2} - 2 \quad \text{by the definition of } L_n.
\end{aligned}$$

Thus, the statement is valid for $n+1$, and by induction, for all $n \geq 1$.

6. Determine if each statement is true or false.

(a)	TRUE	FALSE	$\log_2 n + n^2$ is $\Theta(n^2)$.
(b)	TRUE	FALSE	$\log_2 n + n^2$ is $O(n^2)$.
(c)	TRUE	FALSE	A $\Theta(2^n)$ algorithm is $O(n^3)$.
(d)	TRUE	FALSE	An $O(\sqrt{n})$ algorithm is $\Theta(n)$.
(e)	TRUE	FALSE	A $\Theta(3n)$ algorithm is $O(n)$.
(f)	TRUE	FALSE	A $\Theta(2^n)$ algorithm is $O(3^n)$.

7. Compute $\sum_{n=1}^{\infty} \frac{2^{3n}}{9^n}$. $= \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n$

$$\frac{a}{1-r} = \frac{8/9}{1-8/9} = \frac{8/9}{1/9} = 8$$

Extra Credit: Using mathematical induction, prove that $P_n(x) = (x+1)^n - x^n$ is the solution to the system

$$P_{n+1}(x) = (x+1)P_n(x) + x^n, \quad P_0(x) = 0.$$

To see the base case, note that $P_0(x) = (x+1)^0 - x^0 = 0$.
Now assume the statement is valid for n. Then

$$\begin{aligned} P_{n+1}(x) &= (x+1)P_n(x) + x^n \\ &= (x+1)\left[(x+1)^n - x^n\right] + x^n \quad \text{by assumption} \\ &= (x+1)^{n+1} - x^{n+1} - x^n + x^n \\ &= (x+1)^{n+1} - x^{n+1}. \end{aligned}$$

Therefore, the statement is valid for $n+1$, and by induction, for all $n \in \mathbb{N}$.