1. Solve the linked system $F_{n+1} = F_n + 3G_n$, $G_{n+1} = 4F_n + 2G_n$, $F_0 = 0$, $G_0 = 1$.

$$F_{n+2} = F_{n+1} + 3G_{n+1} = F_{n+1} + 3(4F_n + 2G_n)$$
$$= F_{n+1} + 12F_n + 6G_n = F_{n+1} + 12F_n + 2(F_{n+1} - F_n)$$
$$= 3F_{n+1} + 10F_n.$$

$$\lambda^2 = 3\lambda + 10 \Rightarrow \lambda^2 - 3\lambda - 10 = 0$$
$$(\lambda - 5)(\lambda + 2) = 0$$
$$\lambda = 5, -2.$$

n	0	1
F_n	0	3
G_n	1	2

$$F_n = c_1 \cdot 5^n + c_2(-2)^n$$

$n=0$: $c_1 + c_2 = 0$ $\quad c_2 = -c_1 \Rightarrow 5c_1 + 2c_1 = 3$

$n=1$: $5c_1 - 2c_2 = 3$ $\quad c_1 = \frac{3}{7} \quad c_2 = \frac{-3}{7}$

$$F_n = \frac{3}{7}(5^n - (-2)^n)$$

$$G_n = c_1 \cdot 5^n + c_2(-2)^n$$

$n=0$: $c_1 + c_2 = 1$ $\quad c_2 = 1 - c_1$

$n=1$: $5c_1 - 2c_2 = 2$ $\quad 5c_1 - 2(1 - c_1) = 2$

$$7c_1 = 4$$
$$c_1 = \frac{4}{7} \quad c_2 = \frac{3}{7}$$

$$G_n = \frac{4}{7} \cdot 5^n + \frac{3}{7}(-2)^n$$
$$= \frac{1}{7}(4 \cdot 5^n + 3(-2)^n)$$

2. Solve the recurrence $F_{n+2} = 8F_{n+1} - 16F_n$, $F_0 = 1$, $F_1 = 2$.

$\lambda^2 = 8\lambda - 16$

$\lambda^2 - 8\lambda + 16 = 0$

$(\lambda - 4)^2 = 0 \Rightarrow \lambda = 4$

$F_n = (c_1 + c_2 n)\, 4^n$

$n = 0: \quad c_1 = 1$

$n = 1: \quad (c_1 + c_2) \cdot 4 = 2$

$4 + 4c_2 = 2$

$4c_2 = -2$

$c_2 = -\frac{1}{2}$

$F_n = (1 - \frac{1}{2}n) \cdot 4^n$

$= (1 - \frac{1}{2}n) \cdot 4 \cdot 4^{n-1}$

$= (4 - 2n) \cdot 4^{n-1}$

Extra Credit: Work backwards! Suppose that $g_n = x^n + y^n$ is a solution to

$$g_{n+2} = Ag_{n+1} + Bg_n, \quad g_0 = C, \quad g_1 = D.$$

Find A, B, C, and D.

$\lambda^2 = A\lambda + B$ must have roots x and y, so

$(\lambda - x)(\lambda - y) = 0$

$\lambda^2 - (x+y)\lambda + xy = 0$

$\lambda^2 = (x+y)\lambda - xy \Rightarrow A = x + y \quad B = -xy$

Also, $g_0 = 2 = C$, $g_1 = x + y = D$.

$A = x + y$

$B = -xy$

$C = 2$

$D = x + y$.