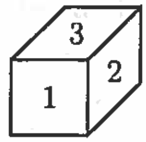
When appropriate, consider this the "start" position. ÷120 for grade.



Please show sufficient steps to your work. This exam is being graded on a point system, and I do not give partial credit for incorrect answers with no supporting work. **ALL WORK MUST BE SHOWN ON THE EXAM PAPER.**

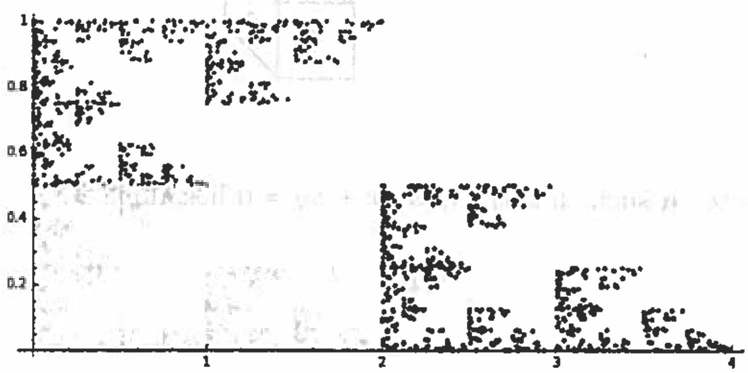
X 4 1. For which values of s are the vectors $\begin{pmatrix} -3 \\ s \end{pmatrix}$ and $\begin{pmatrix} s \\ -27 \end{pmatrix}$ linearly dependent?

$$\frac{-3}{s} = \frac{s}{-27}$$

+2 each

$$s^2 = 81 \Rightarrow s = \pm 9$$

8 2. Write the affine transformations, in matrix form, you would need to create the following fractal. Indicate which transformations correspond to the black, gray, and light gray areas, respectively.



Light gray

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

+2

Gray

$$\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

+2 +1

Black

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

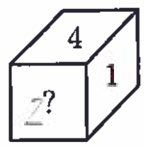
+2 +1

//12

4 3. Find the following matrix product: $\begin{bmatrix} 2 & 4 \\ -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$.

$$\begin{bmatrix} 10 & -6 \\ 3 & -3 \end{bmatrix}$$

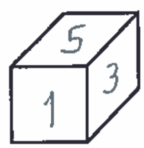
4 4. Write the matrix which transforms the die to the following position:



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

X Less space

3 5. Fill in the die after performing the transformation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.



2 6. Find a vector $\mathbf{u}$ such that the line $7x + 8y = 0$ lies along $\mathbf{u}$.

$(-8, 7)$

5 7. Find the matrix for the projection onto the vector $(-1, 3)$.

$$P\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix}}{\begin{pmatrix} -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \end{pmatrix}} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \frac{-x+3y}{10} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{x-3y}{10} \\ \frac{-3x+9y}{10} \end{pmatrix}$$

+3

$$\begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix}$$

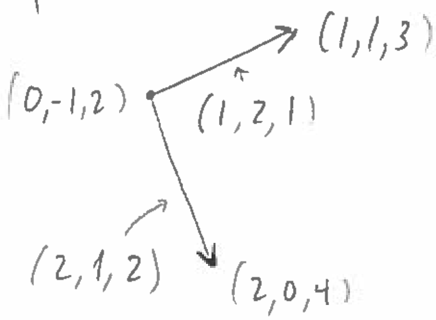
+2

18

$-(2,0,4)$

4 8. Find the area of the triangle with vertices $(0,-1,2)$, $(1,1,3)$, and $(2,0,4)$. $(-2,-1,-2)\times(-1,1,-1)$

$(-3,0,-3)$



$(1,2,1)\times(2,1,2) = (3,0,-3)$

$\frac{1}{2}\|(3,0,-3)\| = \frac{1}{2}\sqrt{9+9} = \frac{1}{2}\sqrt{18}$

$= \frac{3}{2}\sqrt{2}$

-2: correct approach

4 9. Find all eigenvalues and eigenvectors of $\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$.

$\lambda = 1: \begin{pmatrix}1\\0\\0\end{pmatrix}$ $+2$

yz-plane: $90°$ rotation No eigenvalues / eigenvectors. $+2$

4 10. Find the eigenvalues of $\begin{bmatrix}-3&7\\3&1\end{bmatrix}$. **DO NOT** find the eigenvectors!

$\begin{vmatrix}-3-\lambda & 7\\ 3 & 1-\lambda\end{vmatrix} = (-3-\lambda)(1-\lambda) - 21 = \lambda^2 + 2\lambda - 24 = 0$

$(\lambda+6)(\lambda-4) = 0$

$+2$

$\lambda = -6, 4$

$+2$

4 11. The matrix $\begin{bmatrix}2&2\\7&-3\end{bmatrix}$ has an eigenvalue of $\lambda = -5$. Find a corresponding eigenvector.

$\begin{bmatrix}2-(-5) & 2\\ 7 & -3-(-5)\end{bmatrix} = \begin{bmatrix}7&2\\7&2\end{bmatrix}$

$\begin{bmatrix}7&2\\7&2\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow 7x+2y=0$ $\begin{pmatrix}-2\\7\end{pmatrix}$ or $\begin{pmatrix}2\\-7\end{pmatrix}$

$+3$ $+1$

16

4 12. Find the inverse of the matrix $\begin{bmatrix} 2 & -5 \\ 4 & -9 \end{bmatrix}$.

$$\frac{1}{2}\begin{bmatrix} -9 & 5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -9/2 & 5/2 \\ -2 & 1 \end{bmatrix}$$

3 13. Find all a, b for which the following system has a solution.

$$-y + 4x = a$$
$$-8x + 2y = b$$

$$\begin{aligned} -8x + 2y &= -2a \\ -8x + 2y &= b \end{aligned} \Rightarrow b = -2a$$

9 14. Find the LDU decomposition of $\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$.

$$L^{-1} \to \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 6 \end{bmatrix}$$

$$D^{-1} \to \begin{bmatrix} 1 & 0 \\ 0 & 1/6 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$U^{-1} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

16

6 15. How many three-of-a-kind hands are possible with a 52-card deck (no Joker)? You may leave your answer with ${}_nC_r$ notation, but draw an arrow and say what each term means using a short phrase like "choose the suit," for example.

$$\underbrace{{}_{13}C_1}_{\text{choose rank 3OAK}} \times \underbrace{{}_4C_3}_{\text{choose suit}} \times \underbrace{{}_{12}C_2}_{\text{choose single cards}} \times \underbrace{4^2}_{\text{suits for single cards}}$$

${}_4C_2$

-1

4 if convenient 16. How many straight flushes are possible with a 52-card deck (no Joker)? Write your answer as in the previous problem.

$$\underbrace{10}_{\text{choose 1st card}} \times \underbrace{4}_{\text{choose suit}}$$

36 OK.
(not royal flush)

5 17. How many 3 digit numbers can be formed from the digits 2, 3, 5, 6, 7 and 9 which are divisible by 2 and none of the digits is repeated? You must give an exact integer answer!

$$\underbrace{4}_{\text{hundreds digit}} \times \underbrace{5}_{\text{tens digit}} \times \underbrace{2}_{\text{even number}} = 40$$

15

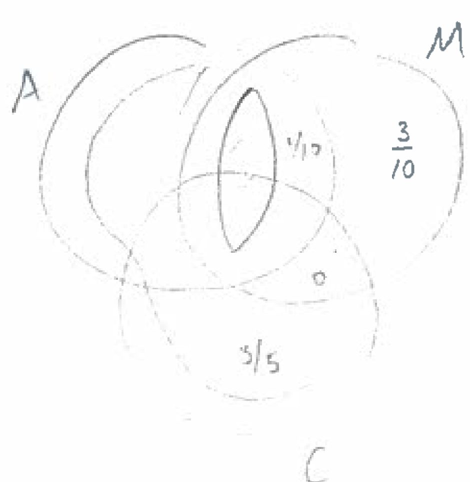
2 18. Suppose a symmetric matrix has eigenvalues $\lambda = 4$ and $\lambda = -7$. An eigenvector corresponding to $\lambda = 4$ is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Give an eigenvector corresponding to $\lambda = -7$.

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

4 19. In how many different ways can the letters of the word DEALING be arranged such that the vowels should always come together? You must give an exact integer answer!

$\underbrace{5!}_{\text{D, L, N, G, vowels}} \times \underbrace{3!}_{\text{ordering for vowels}} = 720$

5 20. At Happyville College, all students take at least one art, math, or CS course. 1/10 of the students take art and math, while 3/5 of the students do not take math. No students take only math and CS. What fraction of the students at Happyville take only math?



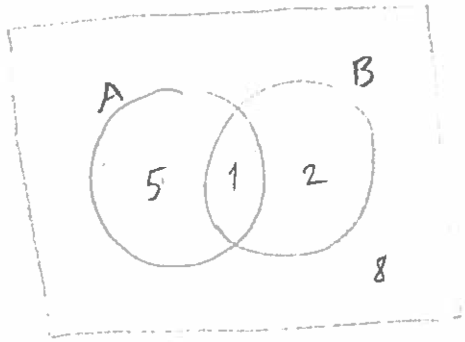
$1 - \frac{1}{10} - \frac{3}{5} = \frac{3}{10}$

11

4

$3/8$ $P(A \cup B) = 1/2$

21. Suppose $P(A) = $ ~~[illegible]~~ $3/8$, $P(A) = 2 \cdot P(B)$, and ~~[illegible]~~ $P(A \cup B) = 1/2$. What is $P(A \cap B)$?



$x + y = \frac{3}{8}$

$x + y = 2(x + \frac{1}{8})$

$y - x = \frac{1}{4}$

$2y = \frac{5}{8}$

$y = \frac{5}{16}$

$x = \frac{1}{16}$

4

22. What is the probability of getting at least two heads in three tosses of a fair coin?

2 heads $\frac{3}{8}$ +2

3 heads $\frac{1}{8}$ +1

$\frac{1}{2}$ +1

6

23. A coin has a 1/3 probability of coming up heads, and a 2/3 probability of coming up tails. What is the probabililty you get at least one head in two flips of the coin? Use generating functions to answer this question!

$$\underbrace{\left(\frac{2}{3}x^0 + \frac{1}{3}x^1\right)^2}_{+\cancel{4}\ 3} = \frac{4}{9}x^0 + \frac{4}{9}x^1 + \frac{1}{9}x^2 \quad +1$$

$\frac{5}{9}$

+1

14

4

24. If B is uniformly distributed in $[0,1]$, find the probability that $\left|B - \frac{2}{3}\right| \le \frac{1}{2}$.

$-\frac{1}{2} \le B - \frac{2}{3} \le \frac{1}{2}$

$+\frac{2}{3} \quad +\frac{2}{3} \quad +\frac{2}{3}$

$\frac{1}{6} \le B \le \frac{7}{6}$

+3

$\frac{1}{6}$

$\frac{5}{6}$

6

25. You randomly break a stick which is 10cm long into two pieces. What is the probability that the difference in the length of the sticks is less than 1cm?

0 $\quad x \quad$ 10

$x \quad 10-x$

$|x-(10-x)| < 1$

$|2x-10| < 1$

$-1 < 2x - 10 < 1$

$9 < 2x < 11$

$\frac{9}{2} < x < \frac{11}{2}$

+3 for trying

$\frac{1}{10}$

6

26. The board below is 20cm on each side, and each strip of black or white is 4cm wide. You toss a coin with diameter 2cm onto the board, and it lies entirely within the 20cm square. What is the probability that the coin touches only one color? In other words, it cannot cross a line between a white and a black strip. (Be careful about determining the sample space!)

$\frac{1}{2}$

2

4

center of coin must be in middle of strip

Harder than I thought!

+4 for a good try

$\frac{36 \cdot 5}{18 \cdot 18} = \frac{180}{324} = \frac{5}{9}$

$\frac{1}{2}$: -1

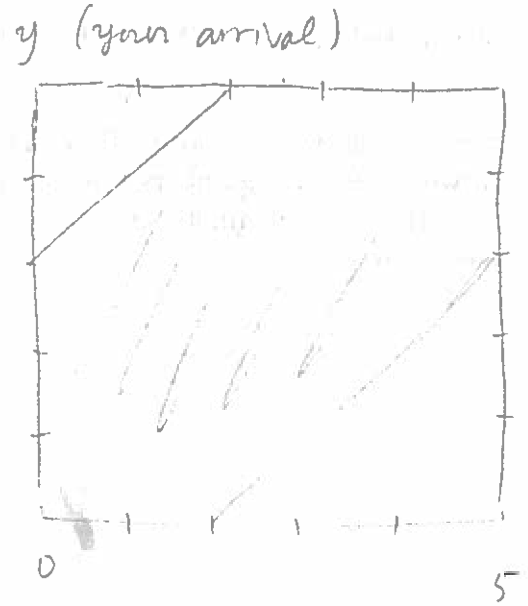
$\frac{180}{400}$: -1

$\frac{9}{20}$

16

8

27. Suppose you and your friend arrived at Fisherman's Wharf last Saturday sometime between 1:00 and 6:00. You stayed exactly two hours, and your friend stayed exactly three hours. Neither of you knew that the other was there. Assuming a uniformly distributed time of arrival, what is the probability that at some time, you were both at Fisherman's Wharf last Saturday?



you arrive before friend leaves

$y \le x + 3$

your friend arrives before you leave

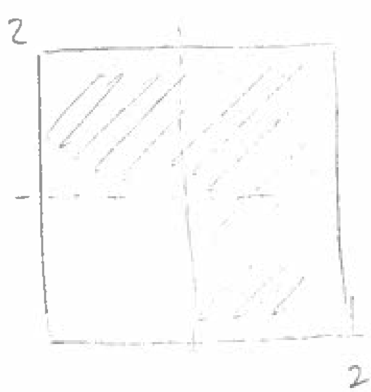
$\left.\begin{matrix} x \le y + 2 \\ y \ge x - 2 \end{matrix}\right\} +5$

$25 - 2 - \frac{9}{2} = \frac{50 - 4 - 9}{2} = \frac{37}{2}$

$\frac{37/2}{25} = \frac{37}{50}$

4

28. Suppose that B and C are uniformly distributed on $[0, 2]$. Find the probability that $\max\{B, C\} > 1$.



$\frac{1}{4} = -1$

$\frac{3}{4}$

12

EXTRA CREDIT:

+3 1. You are playing a game with dice where every time you roll a die, you get the number of points shown on the die – except when you roll a 5, you *lose* 5 points. For all other numbers, you gain points. Using generating functions, explain how you would find the probability of having a total of 3 points after four rolls of the die.

+3 2. Suppose that A, B, and C are uniformly distributed in $[0, 1]$. What is the probability that $A + B + C < 1$?

+3 3. You have 20 cards, each with a number from 1 to 20 written on it. How many ways are there to choose five cards such that no two cards have numbers that are just one apart? In other words, if one of your cards is the 8, you cannot have the 7 or the 9. Use ${}_nC_r$ notation; you do not need an integer answer.