+10 1. Write a *matrix* equation for solving the following system of equations, and then find all solutions to this system. Show your steps. No points for solving any other way!

$$-7x + 3 = 4y, \qquad 2y + 3x = 1.$$

$$-7x + 4y = 3$$
$$3x + 2y = 1$$

$$\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix} \Rightarrow \frac{1}{2} \begin{bmatrix} 2 & -4 \\ -3 & 7 \end{bmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

+5 2. In two dimensions, it is possible for a matrix to have no real eigenvalues or eigenvectors. First, give an example of this. Second, briefly explain why *every* linear transformation in three dimensions must have at least one real eigenvalue.

In 3D, the characteristic equation is a cubic equation. Every cubic equation has at least one real root, so every linear transformation in 3D has at least one real eigenvalue.

+10 3. Using your die, find all eigenvalues and eigenvectors of $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

$\lambda = 1$: $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = -1$: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ or any 2 linearly independent vectors in the plane $y=0$.

+5 4. For what choices of u and v does the following system have solution(s)?

$\times -5$ $$2x - 3y = u, \qquad -10x + 15y = v.$$

$-10x + 15y = -5u.$

In order to have solutions, these lines must be coincident (and not parallel). So we must have $-5u = v$.

+10 5. Find all eigenvalues of $\begin{bmatrix} 4 & -12 \\ -9/2 & 1 \end{bmatrix}$. Show all steps!

$$\det \begin{bmatrix} 4-\lambda & -12 \\ -9/2 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) - 54 = 0$$

$$\lambda^2 - 5\lambda - 50 = 0$$

$$(\lambda - 10)(\lambda + 5) = 0$$

$$\lambda = 10, -5$$

+10 6. $A = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ has an eigenvalue of 2. Find the corresponding eigenvector(s).

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} x + 2y - 2z = 2x \\ y = 2y \\ -x + 2y = 2z \end{matrix} \Rightarrow y = 0$$

From $y = 0$, we get $x = -2z$ (in either the 1^{st} or 3^{rd} equation)

Eigenvector $= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ or some parallel vector

+5 Extra Credit: Let A be a linear transformation with eigenvalues λ_1 and λ_2, and let $k \neq 0$ be a constant. If $B = kA$, what are the eigvenvalues of B? Show your work!

Suppose λ is an eigenvalue for A, with eigenvector $\underline{v}$. Then $A\underline{v} = \lambda \underline{v}$. Then $B\underline{v} = kA\underline{v} = k\lambda\underline{v}$, and so $k\lambda$ is an eigenvalue of B. Thus, the eigenvalues of B are k times the eigenvalues of A.