Making Calculus Accessible

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Contents

Chapter 1

Preliminaries

1.1 Algebra and Trigonometry Review

You won't need everything you learned in Precalculus, but there are several topics which will be important. Review by working through the following problems.

Review Problems

1. Find an equation of a line in the form $y = mx + b$ for a line with slope -2 which passes through the point $(1, -6)$.
2. Find an equation of a line in the form $y = mx + b$ for a line passing through points $(3, -5)$ and $(-2, 1)$.
3. Simplify by factoring the numerator: $\dfrac{x^2 - x - 6}{x - 3}$.
4. Simplify by factoring the numerator: $\dfrac{2x^2 + 3x - 5}{x - 1}$.
5. Rationalize the denominator: $\dfrac{3}{\sqrt{7} - 2}$.
6. Rationalize the numerator: $\dfrac{\sqrt{x} - 1}{x - 1}$.
7. Rationalize the numerator: $\dfrac{\sqrt{x} - \sqrt{a}}{x - a}$.
8. Simplify: $(\sqrt{x})^4$.
9. Expand, combining exponents: $x^2(x + \sqrt{x})$.
10. Write using negative exponents: $\dfrac{2}{x^4} - \dfrac{3}{\sqrt{x}}$.
11. Write using positive exponents in the denominator: $x^{-1/3} + x^{-5}$.
12. Expand, combining exponents: $x^3\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$.
13. Factor out x^2 from the following expression: $3x^7 + x^{7/2}$.
14. Factor out $x^{3/2}$ from the following expression: $2x^4 - x^{3/2}$.
15. Add by finding a common denominator: $\dfrac{2}{5} + \dfrac{7}{6}$
16. Simplify using a common denominator: $\dfrac{1}{2x} - \dfrac{2}{y}$
17. Simplify: $\dfrac{\frac{1}{2} + \frac{1}{x}}{x}$

18. Simplify: $\dfrac{\dfrac{1}{3}-\dfrac{1}{y}}{3-y}$

19. Convert from degrees to radians:

    (a) 60°

    (b) 180°

    (c) -210°

20. Convert from radians to degrees:

    (a) $\dfrac{3\pi}{4}$

    (b) $-\dfrac{\pi}{2}$

    (c) $\dfrac{5\pi}{3}$

21. Evaluate the following, giving exact answers.

    (a) $\cos 0$

    (b) $\sin 90^\circ$

    (c) $\tan \dfrac{\pi}{3}$

    (d) $\sin \dfrac{5\pi}{4}$

    (e) $\cos 300^\circ$

    (f) $\tan \dfrac{3\pi}{2}$

Assessment Expectations: For algebra problems, any like those above. For the Unit Circle, you will be given a blank Unit Circle, with questions like:

1. What angle in degrees corresponds to Point A?
2. What angle in radians corresponds to Point B?
3. What are the coordinates of Point C?
4. What is $\sin(60°)$? Can be cos or tan, and the angle may be in radians.

Solutions

1.

$$\begin{aligned} y &= mx + b \\ y &= -2x + b \\ -6 &= -2(1) + b \\ -4 &= b \\ y &= -2x - 4 \end{aligned}$$

2. To find the slope, use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, with $(x_1, y_1) = (3, -5)$ and $(x_2, y_2) = (-2, 1)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-5)}{-2 - 3} \\ &= -\frac{6}{5} \end{aligned}$$

Now use $y = mx + b$ with the point $(3, -5)$.

$$\begin{aligned} y &= -\frac{6}{5}x + b \\ -5 &= -\frac{6}{5}(3) + b \\ -\frac{25}{5} + \frac{18}{5} &= b \\ -\frac{7}{5} &= b \\ y &= -\frac{6}{5}x - \frac{7}{5} \end{aligned}$$

3.

$$\begin{aligned} \frac{x^2 - x - 6}{x - 3} &= \frac{(x - 3)(x + 2)}{x - 3} \\ &= x + 2 \end{aligned}$$

4.

$$\begin{aligned}\frac{2x^2+3x-5}{x-1} &= \frac{(x-1)(2x+5)}{x-1} \\ &= 2x+5\end{aligned}$$

5.

$$\begin{aligned}\frac{3}{\sqrt{7}-2} &= \frac{3}{\sqrt{7}-2}\cdot\frac{\sqrt{7}+2}{\sqrt{7}+2} \\ &= \frac{3(\sqrt{7}+2)}{7+2\sqrt{7}-2\sqrt{7}-4} \\ &= \frac{3(\sqrt{7}+2)}{3} \\ &= \sqrt{7}+2\end{aligned}$$

6.

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{\sqrt{x}-1}{x-1}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1} \\ &= \frac{x+\sqrt{x}-\sqrt{x}-1}{(x-1)(\sqrt{x}+1)} \\ &= \frac{x-1}{(x-1)(\sqrt{x}+1)} \\ &= \frac{1}{\sqrt{x}+1}\end{aligned}$$

7.

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{a}}{x-a} &= \frac{\sqrt{x}-\sqrt{a}}{x-a}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} \\ &= \frac{x+\sqrt{x}\sqrt{a}-\sqrt{x}\sqrt{a}-a}{(x-a)(\sqrt{x}+\sqrt{a})} \\ &= \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} \\ &= \frac{1}{\sqrt{x}+\sqrt{a}}\end{aligned}$$

8.

$$\begin{aligned}(\sqrt{x})^4 &= \left(x^{1/2}\right)^4 \\ &= x^{(1/2)4} \\ &= x^2\end{aligned}$$

9.

$$\begin{aligned}x^2(x+\sqrt{x}) &= x^2(x+x^{1/2}) \\ &= x^2\cdot x + x^2\cdot x^{1/2} \\ &= x^3+x^{5/2}\end{aligned}$$

10.
$$\frac{2}{x^4} - \frac{3}{\sqrt{x}} = 2x^{-4} - 3x^{-1/2}$$

11.
$$x^{-1/3} + x^{-5} = \frac{1}{\sqrt[3]{x}} + \frac{1}{x^5}$$

12.
$$\begin{aligned} x^3\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) &= x^3(x^{1/2} + x^{-1/2}) \\ &= x^3 \cdot x^{1/2} + x^3 \cdot x^{-1/2} \\ &= x^{7/2} + x^{5/2} \end{aligned}$$

13.
$$\begin{aligned} 3x^7 + x^{7/2} &= x^2(3x^{7-2} + x^{7/2-2}) \\ &= x^2(3x^5 + x^{3/2}) \end{aligned}$$

14.
$$\begin{aligned} 2x^4 - x^{3/2} &= x^{3/2}(2x^{4-3/2} - x^{3/2-3/2}) \\ &= x^{3/2}(2x^{5/2} - 1) \end{aligned}$$

15.
$$\begin{aligned} \frac{2}{5} + \frac{7}{6} &= \frac{2}{5} \cdot \frac{6}{6} + \frac{7}{6} \cdot \frac{5}{5} \\ &= \frac{12}{30} + \frac{35}{30} \\ &= \frac{47}{30} \end{aligned}$$

16.
$$\begin{aligned} \frac{1}{2x} - \frac{2}{y} &= \frac{1}{2x} \cdot \frac{y}{y} - \frac{2}{y} \cdot \frac{2x}{2x} \\ &= \frac{y}{2xy} - \frac{4x}{2xy} \\ &= \frac{y - 4x}{2xy} \end{aligned}$$

17. Method 1: First, combine the numerator.
$$\begin{aligned} \frac{1}{2} + \frac{1}{x} &= \frac{1}{2} \cdot \frac{x}{x} + \frac{1}{x} \cdot \frac{2}{2} \\ &= \frac{x + 2}{2x} \end{aligned}$$

Then multipliy by the reciprocal of the denominator.

$$\begin{aligned}\frac{\frac{x+2}{2x}}{x} &= \frac{x+2}{2x}\cdot\frac{1}{x}\\ &= \frac{x+2}{2x^2}\end{aligned}$$

Method 2: The least common denominator in the fractions is $2x$. Multiply top and bottom by $2x$.

$$\begin{aligned}\frac{\frac{1}{2}+\frac{1}{x}}{x} &= \frac{\frac{1}{2}+\frac{1}{x}}{x}\cdot\frac{2x}{2x}\\ &= \frac{\frac{1}{2}\cdot 2x+\frac{1}{x}\cdot 2x}{x\cdot 2x}\\ &= \frac{x+2}{2x^2}\end{aligned}$$

18. Method 1: First, combine the numerator.

$$\begin{aligned}\frac{1}{3}-\frac{1}{y} &= \frac{1}{3}\cdot\frac{y}{y}-\frac{1}{y}\cdot\frac{3}{3}\\ &= \frac{y-3}{3y}\end{aligned}$$

Then multipliy by the reciprocal of the denominator.

$$\begin{aligned}\frac{\frac{y-3}{3y}}{y} &= \frac{y-3}{3y}\cdot\frac{1}{3-y}\\ &= \frac{y-3}{3y(3-y)}\\ &= \frac{-(3-y)}{3y(3-y)}\\ &= -\frac{1}{3y}\end{aligned}$$

Method 2: The least common denominator in the fractions is $3y$. Multiply top and bottom by $3y$.

$$\begin{aligned}\frac{\frac{1}{3}-\frac{1}{y}}{y} &= \frac{\frac{1}{3}-\frac{1}{y}}{3-y}\cdot\frac{3y}{3y}\\ &= \frac{\frac{1}{3}\cdot 3y-\frac{1}{y}\cdot 3y}{(3-y)\cdot 3y}\\ &= \frac{y-3}{(3-y)\cdot 3y}\\ &= \frac{-(3-y)}{(3-y)\cdot 3y}\\ &= -\frac{1}{3y}\end{aligned}$$

19. Use the conversion factor 1 degree $= \dfrac{\pi}{180}$ radians.

    (a) $60 \cdot \dfrac{\pi}{180} = \dfrac{\pi}{3}$

    (b) $180 \cdot \dfrac{\pi}{180} = \pi$

    (c) $-210 \cdot \dfrac{\pi}{180} = -\dfrac{7\pi}{6}$

20. Use the converstion factor 1 radian $= \dfrac{180}{\pi}$ degrees.

    (a) $\dfrac{3\pi}{4} \cdot \dfrac{180}{\pi} = 135°$

    (b) $-\dfrac{\pi}{2} \cdot \dfrac{180}{\pi} = -90°$

    (c) $\dfrac{5\pi}{3} \cdot \dfrac{180}{\pi} = 300°$

21. Use the unit circle. If $\cos\theta = 0$, then $\tan\theta$ is undefined. Otherwise, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

    (a) 1

    (b) 1

    (c) $\sqrt{3}$

    (d) $-\dfrac{1}{\sqrt{2}}$

    (e) $\dfrac{1}{2}$

    (f) Undefined

1.2 Introduction to Physics

Much of calculus was developed to study physics. While a course in physics is not required for calculus, there are a few fundamental concepts from physics that we will use over and over again. This is a summary of those concepts.

Most of us are familiar with driving a car. The speedometer measures the **speed** at which you're traveling, while the odometer measures the **distance** you've traveled.

Let's suppose you take a two-hour drive, and you drive at a constant rate of 30 km/hr. Below is a graph of your starting position, at $x = 0$. What is your ending position?

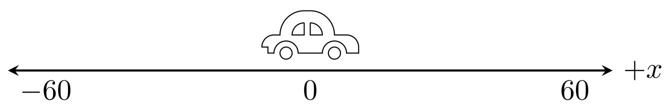


Figure 1.1: Starting position.

If you drove east for two hours (we'll describe going in the positive direction as going east, and going in the negative direction as going west), you'd be 60 km east of where you began.

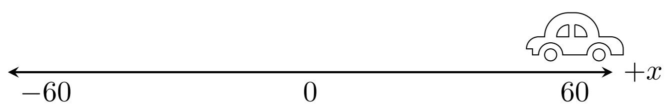


Figure 1.2: Ending position A.

But if you drove west for two hours, you'd be 60 km west of where you began.

Figure 1.3: Ending position B.

But maybe you drove east for one hour, and then turned around and drove west for an hour. Then you'd be right back where you started.

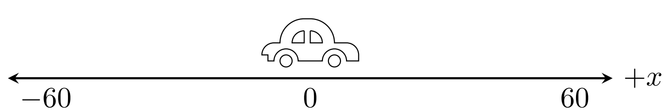


Figure 1.4: Ending position C.

Since you're driving at a constant speed, your speed graph would look like this.

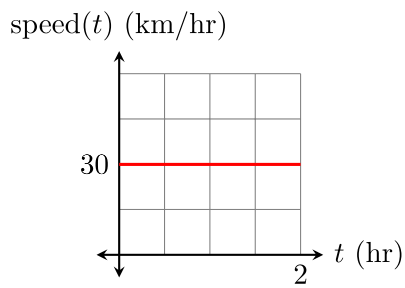


Figure 1.5: Speed graph.

The main issue is this: you can't know your ending position by looking at the speed graph. There is not enough information. That is why the concept of **velocity** is so important in science. Essentially, velocity is speed *and* direction. When you're driving east, your speed and velocity are both 30 km/hr. But when you're driving west, your speed is 30 km/hr, but your velocity is -30 km/hr.

What about a graph of the distance you traveled? Since distance equals rate times time and you're driving at a constant speed, you've traveled $30 \times t$ km in t hours. So the distance traveled up to time t is represented by the graph below.

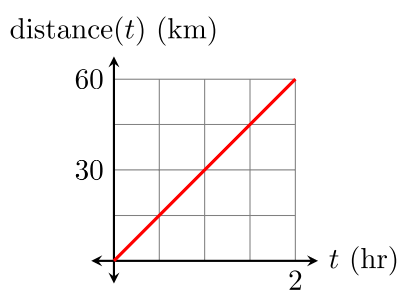


Figure 1.6: distance(t) $= 30t$

This graph tells you that you've driven a total of 60 km, but there is no way to know *where* you ended up. Where you end up relative to where you began is called **displacement** in physics. Similar to how velocity is speed with a direction, displacement is *distance* with a direction. So your displacement in Ending position A is 60 km, but your displacement in Ending position B is -60 km, since you ended up 60 km west of where you started. Your displacement in Ending position C is 0 km, since you're back where you started.

Ending position A

Let's see how we apply the concepts of velocity and displacement to each of the three scenarios described above. We'll redraw Figure 1.2 with an arrow representing the path taken.

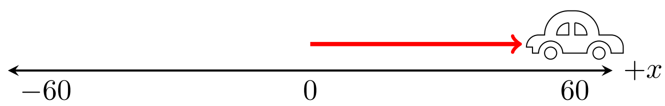


Figure 1.7: Ending position A.

In this scenario, speed and velocity are the same: 30 km/hr for two hours. The letter "v" is used in physics to represent velocity. This is shown on the left of Figure 1.8.

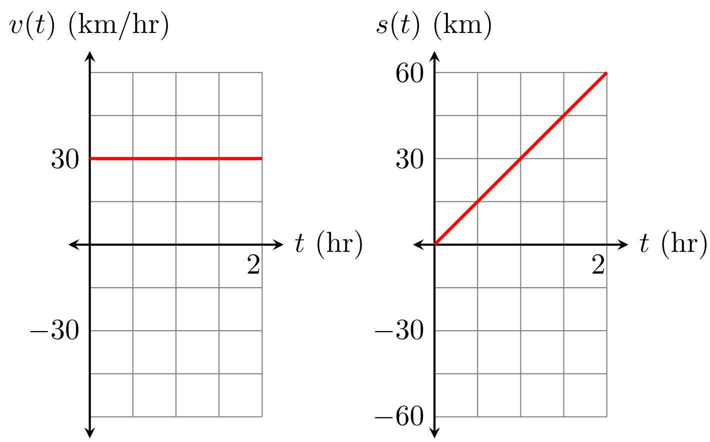


Figure 1.8: Velocity and displacement graphs for ending position A.

In this case, the distance is the same as the displacement, so the displacement graph is identical to Figure 1.6. We use the letter "s" for displacment, since the letter "d" is used for describing derivatives in calculus. Notice that the velocity is positive here, and the slope of the displacement is positive. This is not a coincidence, and is another concept we'll be exploring in calculus.

Ending position B

Let's redraw Figure 1.3 to reflect the path taken.

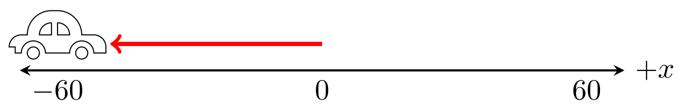


Figure 1.9: Ending position B.

In this case, you drove west for two hours, and so your velocity is -30 km/hr for two hours. This is shown on the left of Figure 1.10. Because your velocity is negative, you end up 60 km west from where you started: a total displacement of -60 km. Notice that the displacement graph has a negative slope because the velocity is negative. This is shown on the right of Figure 1.10.

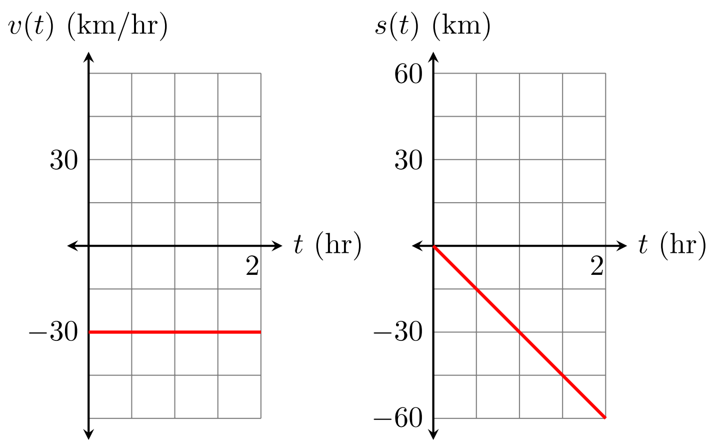


Figure 1.10: Velocity and displacement graphs for ending position B.

Ending position C

Here, Figure 1.4 is redrawn to include the path taken.

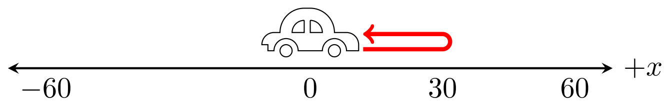


Figure 1.11: Ending position C.

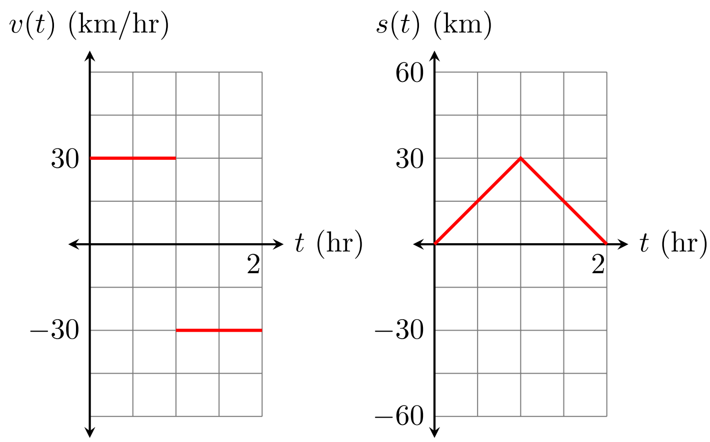


Figure 1.12: Velocity and displacement graphs for ending position C.

Here, you drove east for one hour (30 km/hr) and west for the next hour (−30 km/hr). So the velocity curve jumps down to −30 after one hour.

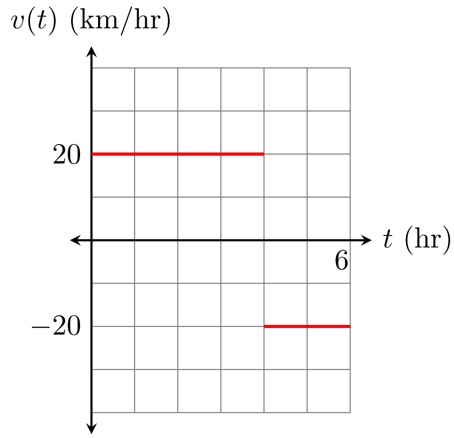
But for ending position C, you start driving east, so the displacement graph is sloping upward. But after an hour – since you turned around – the displacement graph begins sloping downward, so by the time two hours have gone by, your displacement is 0 km, since you ended where you started.

The important point is this: if I gave you one of the *velocity* graphs for any of the three scenarios, you could tell me *exactly* where I ended up. But all three ending positions have the *same* speed graph (shown in Figure 1.5). So in physics and science, "velocity" is a much more useful concept than "speed."

Also, if I gave you one of the displacement graphs for any of these scenarios, you could tell me *exactly* what my trip looked like and where I ended up. But if I just gave you the distance graph (as shown in Figure 1.6), the *only* thing you could tell me was that I drove 60 km. There would not be enough information to conclude any more about the nature of my trip.

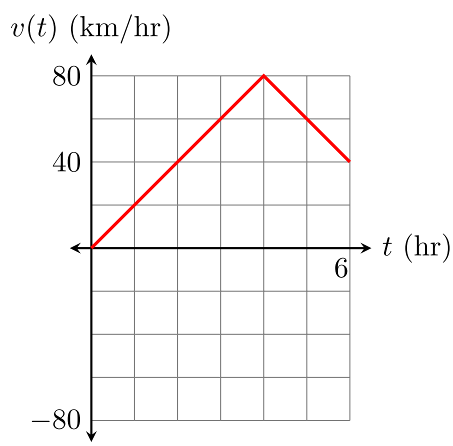
Example 1

Below is the velocity graph for a trip. Describe the journey and draw the corresponding displacement graph.



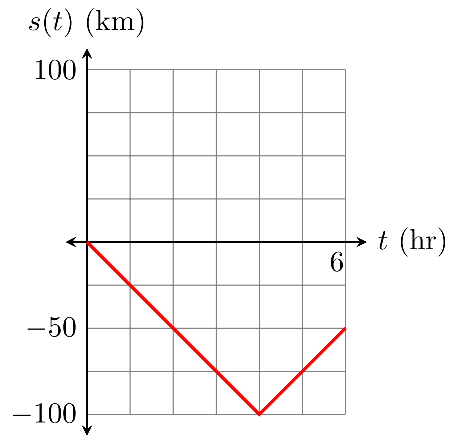
For this journey, we are driving east at 20 km/hr for four hours, then turning around and driving west at 20 km/hr for two more hours. What does the displacement graph look like?

After 4 hours at 20 km/hr, you would have gone 80 km. So this tells you how to draw the first half of the displacement graph. Once you've gone 80 km, you turn around and drive at -20 km/hr for two hours, so you go west 40 km, bringing your net displacement to 40 km. This tells you how to draw the second part of the graph.



Example 2

Below is the displacement graph for a trip. Describe the journey and draw the corresponding velocity graph.



For this journey, we're driving 100 km west, and then 50 km east, so that we end up 50 km west of where we started. What does the velocity graph look like?

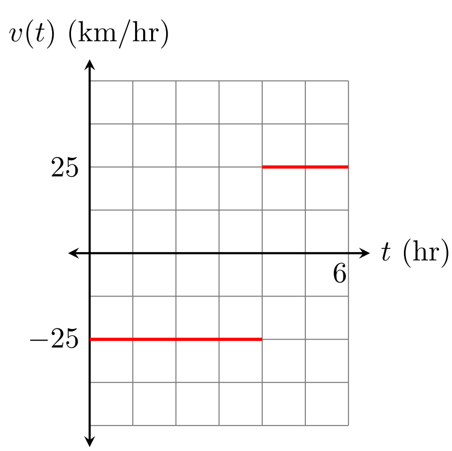
Recall that the velocity is the slope of the displacement graph, so for the first four hours, we get

$$v(t) = \frac{\text{rise}}{\text{run}} = \frac{-100}{4} = -25 \text{ km/hr}.$$

For the last two hours, we get

$$v(t) = \frac{\text{rise}}{\text{run}} = \frac{50}{2} = 25 \text{ km/hr}.$$

Thus, the velocity graph looks like the graph below.

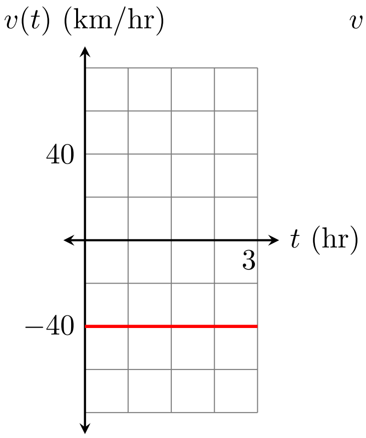


Summary

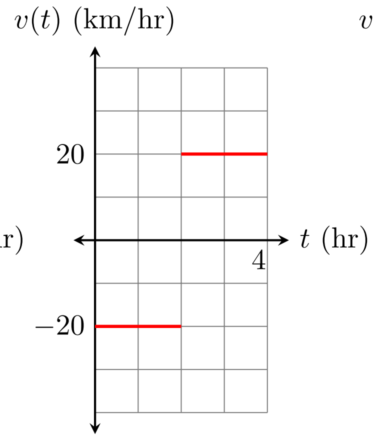
While speed and distance are concepts well-suited to day-to-day life, in the world of physics and science, they are not very precise. Graphs of speed and distance provide very little information about the nature of a journey. However, by introducing the concepts of velocity and displacement, we get an extremely accurate representation of what is actually going on. Essentially, calculus is an in-depth study of the relationship between velocity and displacement graphs.

Homework

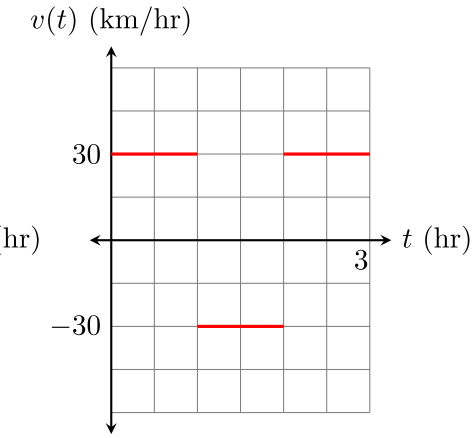
1. Each of the following is the velocity graph of a car trip. For each graph, (1) write a sentence explaining the trip in words, and (2) draw the corresponding displacement graph. Label your graphs carefully!



(a)

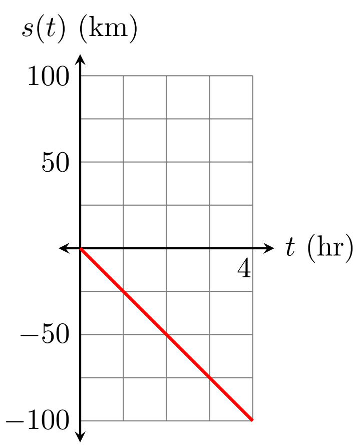


(b)

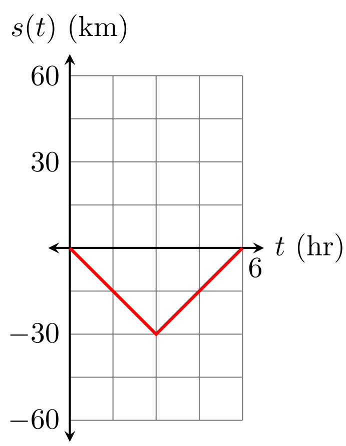


(c)

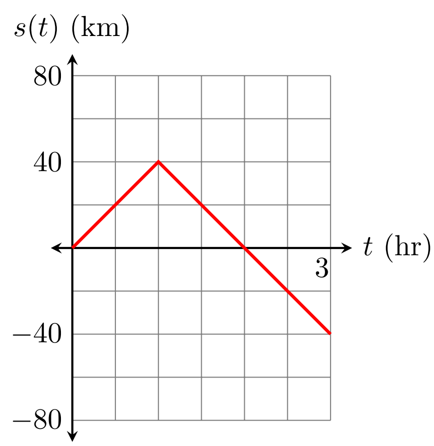
2. Each of the following is the displacement graph of a car trip. For each graph, (1) write a sentence explaining the trip in words, and (2) draw the corresponding velocity graph. Label your graphs carefully!



(a)



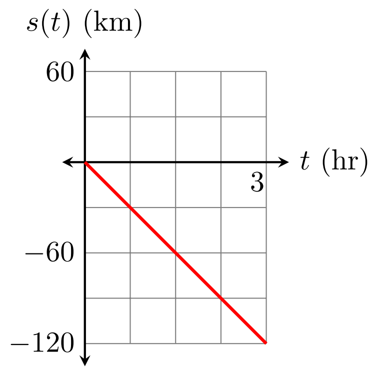
(b)



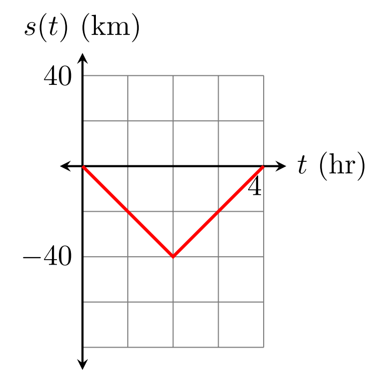
(c)

Solutions

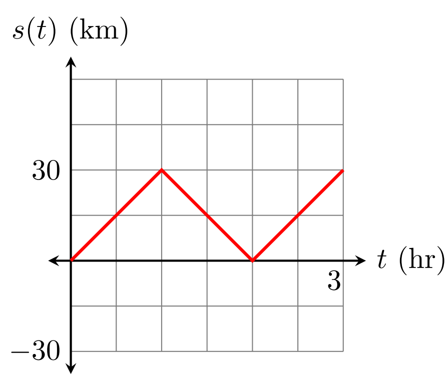
1. (a) You drove 40 km/hr west for three hours.



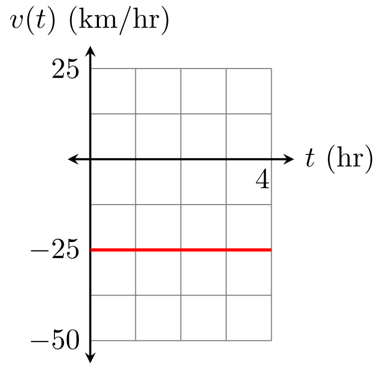
(b) You drove west at 20 km/hr for two hours, and then turned around and drove east at 20 km/hr for two hours.



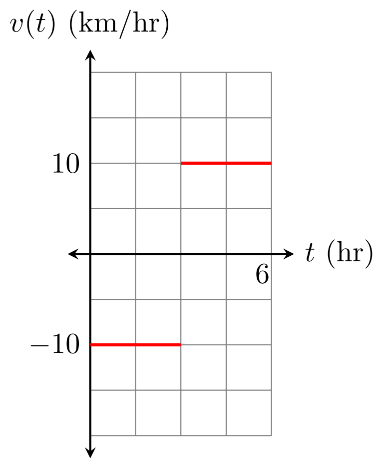
(c) You drove east at 30 km/hr for one hour, turned around and drove west at 30 km/hr for another hour, and then turned around and drove east for one hour at 30 km/hr.



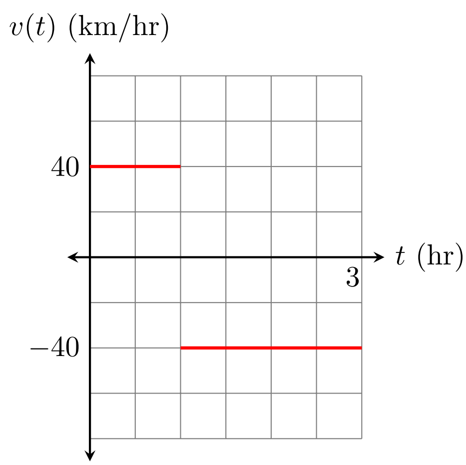
2. (a) You drove 100 km west in four hours, and therefore drove at a velocity of -25 km/hr.



(b) You drove 30 km west in three hours – at a velocity of -10 km/hr, and then drove east 30 km in three hours – at a velocity of 10 km/hr.



(c) You drove 40 km east in one hour (40 km/hr), and then turned around and drove 80 km west in two hours (a velocity of -40 km/hr).



1.3 What Calclulus is All About

We will look at the basic concepts behind calculus by studying three examples. Much of calculus comes from physics, so we'll focus our attention on velocity and displacement. In physics, velocity and displacement can be positive or negative, so we use these terms instead of speed and distance.

Example 1

The graphs below represent a road trip you might go on. You drive at a constant rate of 20 km/hr for 4 hours, and so $v(t) = 20$.

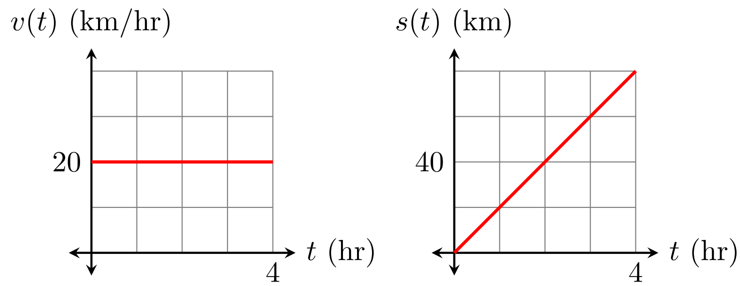


Figure 1.13: Velocity graph (left), and displacement graph (right).

Since displacement = rate × time, then $s(t) = 20t$. This is graphically represented below by the blue rectangle.

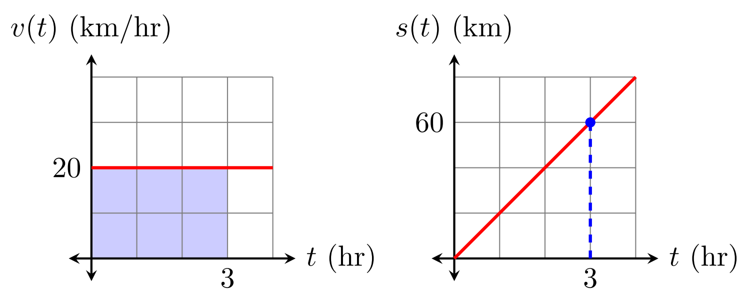


Figure 1.14: Velocity graph (left), and displacement graph (right).

Let's look at what happens after driving for three hours. You've traveled a total of $s(3) = 60$ km, which is represented by the blue rectangle; that is, the area under $v(t)$ up to three hours. Also, the slope of the line at $(3, 60)$ on the displacement curve is $v(3)$, which is 20 km/hr. It might look like the slope is 1, but remember that the units on the axes are different.

This example (and the ones that follow) illustrate this very important principle in physics:

> The area underneath the velocity curve up to time t corresponds to the displacement at time t, and the slope of the displacement curve at time t is the velocity at time t.
>
> Using function notation, we would say that the area up to t underneath $v(t)$ is $s(t)$, and the slope of the tangent line at $(t, s(t))$ is given by $v(t)$.

Example 2

Now we'll take a look at a road trip where your velocity is not constant. Since you are driving at 40 km/hr after 4 hours, your velocity is given by $v(t) = 10t$.

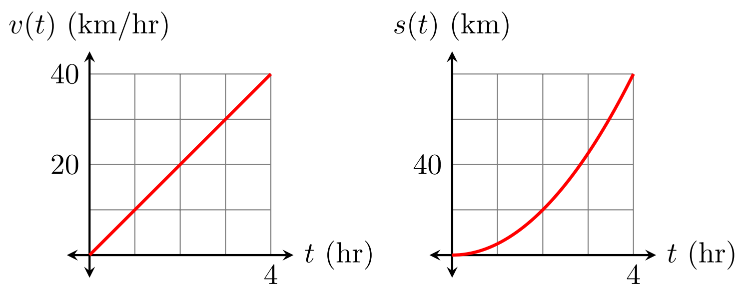


Figure 1.15: Velocity graph (left), and displacement graph (right).

Since your velocity is not constant, we can't use displacement = rate × time. But we can still use the fact that $s(t)$ is the area under $v(t)$ up to t hours, just as in Example 1.

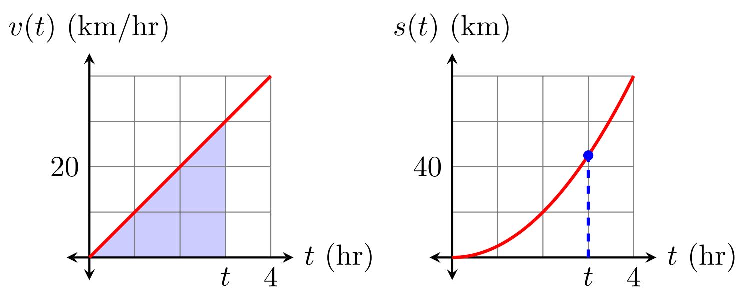


Figure 1.16: Velocity graph (left), and displacement graph (right).

Since the blue area is a triangle, we can use $A = \frac{1}{2}bh$, where $b = t$ is the base and $h = 10t$ is the height. So

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot b \cdot h \\ &= \frac{1}{2} \cdot t \cdot 10t \\ &= 5t^2. \end{aligned}$$

This function is graphed on the right; it is a parabola.

In Example 1, we saw that the slope of the line at $(3, s(3))$ was 20 km/hr. In this example, $s(3) = 45$. But what is the slope of the parabola at $(3, 45)$? Now we've entered calculus territory. We're not only interested in the slope of a *line,* we're interested in the slope of a *curve.*

We can accomplish this by looking at the tangent line to a curve. Below (middle graph), you can see that the blue line intersects the parabola *only* at the point $(3, 45)$. In geometry, when a line intersects a curve at just one point, we call this a **tangent line.** When you zoom in on the blue box (right graph), you see that near the point $(3, 45)$, it's hard to tell the difference between the tangent line and the parabola. This is an important property in calculus.

What is the slope of this tangent line? Since $v(3) = 30$, this tangent line has a slope of 30 km/hr.

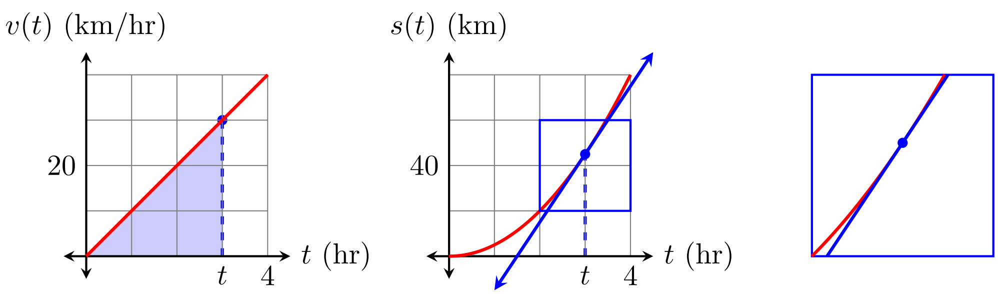


Figure 1.17: Velocity graph (left), tangent line (middle), zooming in (right).

Example 3

Let's look at another road trip. This time, you start out at 20 km/hr, but you slow down at a constant rate. After 2 hours, you turn around and start driving in the opposite direction. This is why we use *velocity* instead of *speed*. If you are driving east with a positive velocity, it means that if you turn around and drive west, your velocity is negative.

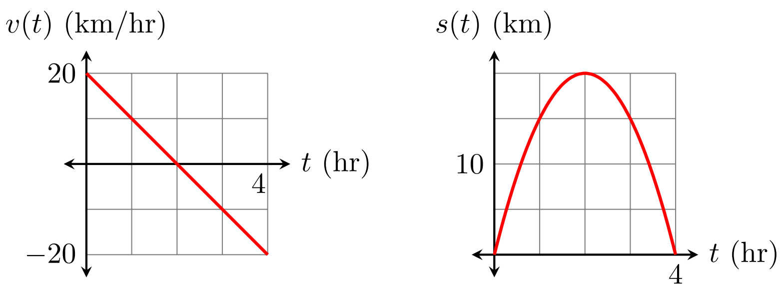


Figure 1.18: Velocity graph (left), displacement graph (right).

Since you start out at 20 km/hr and end up at -20 km/hr after 4 hours (meaning you end up driving in the opposite direction), then $v(t) = 20 - 10t$, which you get by finding the equation of the line between $(0, 20)$ and $(4, -20)$.

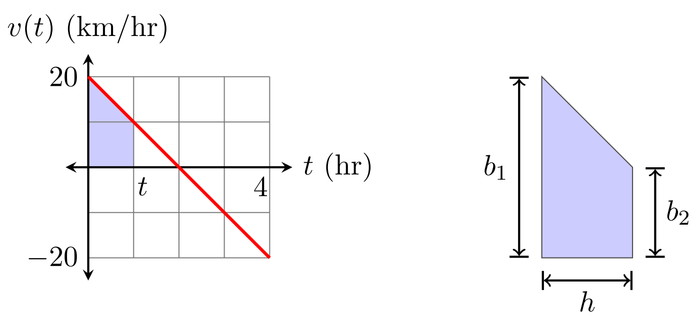


Figure 1.19: Area under the velocity graph (left), area of the trapezoid (enlarged, right).

As in Example 2, we can find the displacement curve by looking at an area. This time, the area is a trapezoid, so we need the formula $A = \frac{1}{2}(b_1 + b_2)h$ from geometry. In our example, b_1 is always 20, b_2 corresponds to $v(t)$, and h is just t. This means that

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot (b_1 + b_2) \cdot h \\ &= \frac{1}{2} \cdot (20 + v(t)) \cdot t \\ &= \frac{1}{2}(20 + (20 - 10t)) \cdot t \\ &= 20t - 5t^2. \end{aligned}$$

Now look at the graph below. Since $v(t)$ is positive here, the slope of the tangent line to the curve $s(t)$ is positive.

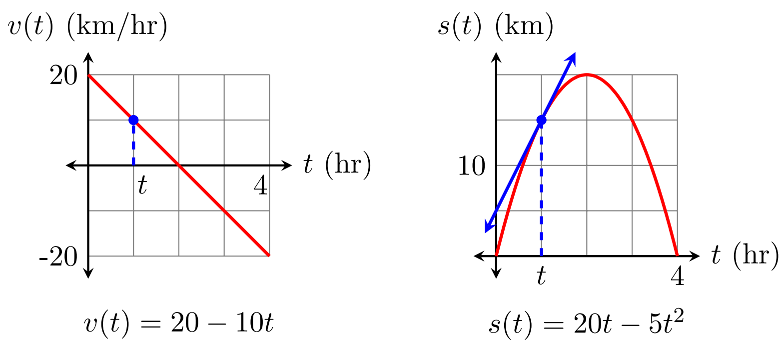


Figure 1.20: Velocity graph (left), displacement graph (right).

But in the next graph, the value of $v(t)$ is negative, and you can see that the slope of the tangent line to the curve is negative.

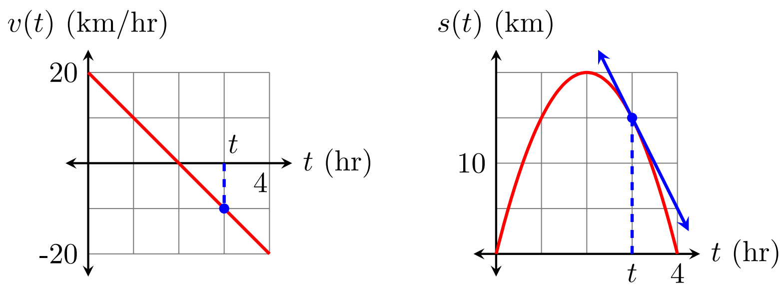


Figure 1.21: Velocity graph (left), displacement graph (right).

Now when $t = 4$, you can see that $s(4) = 0$. This represents the area under the velocity curve up to $t = 4$. How can this area be zero?

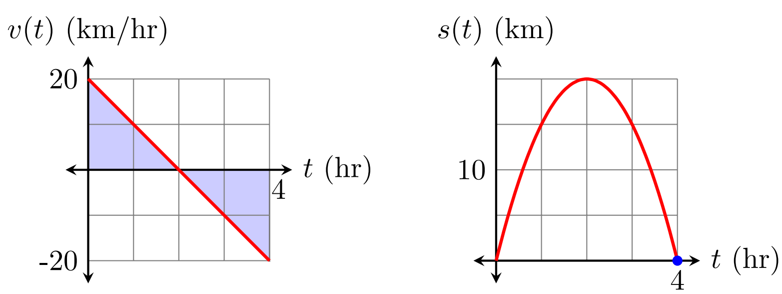


Figure 1.22: Velocity graph (left), displacement graph (right).

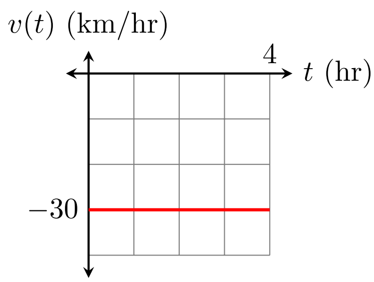
In calculus, areas can be *negative.* Let's say when your velocity is positive, you're driving east, so when your velocity is negative, you're driving west. Now the area *below* the velocity curve when the velocity is *positive* is how far you traveled east. But the area *above* the velocity curve when the velocity is *negative* is how far you traveled west – and this area is *negative.* Looking at the graph on the left, the triangles are congruent, but one has positive area and the other has negative area. Their areas cancel out, which is why your total displacement is 0 – you've traveled just as far while driving east as you did while driving west.

So considering velocity and displacement, instead of speed of distance, is very important in calculus.

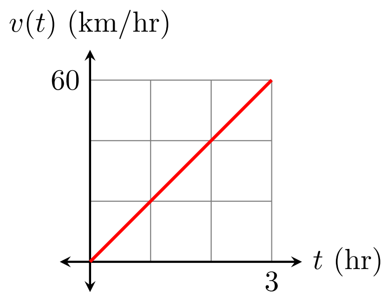
> When your velocity is positive, the corresponding slope of the tangent line on the displacement curve will be positive.
>
> When your velocity is negative, you're traveling in the opposite direction, and the corresponding slope of the tangent line on the displacement curve will be negative.
>
> The area corresponding to traveling with a positive velocity will always be positive, so your displacement will be positive.
>
> The area corresponding to traveling with a negative velocity in the opposite direction will always be negative, so your displacement will be negative.

Homework

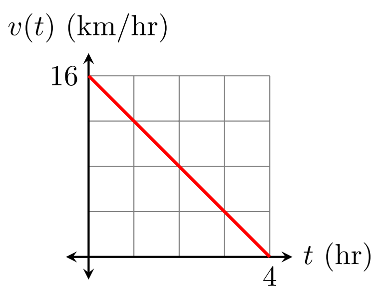
1. Below is a graph of a velocity curve. Find an equation for the displacement curve.



2. Below is a graph of a velocity curve. Find an equation for the displacement curve.



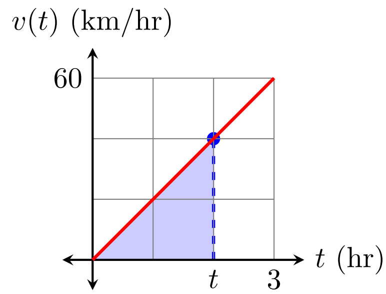
3. Below is a graph of a velocity curve. Find an equation for the displacement curve.



Solutions

1. Since the velocity is constant, we find displacement by multiplying the velocity by time, and so $s(t) = -30t$.

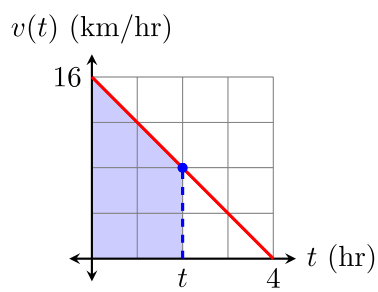
2. This problem is similar to Example 2.



Since we are at 60 km/hr after 3 hr, the slope of the line must be $\frac{60}{3} = 20$. Since the line goes through the origin, the equation of the line is $v(t) = 20t$. Finding the area of the triangle shown above (just as in Example 2), we get

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot b \cdot h \\ &= \frac{1}{2} \cdot t \cdot 20t \\ &= 10t^2. \end{aligned}$$

3. This problem is similar to Example 3.



To find the area of the trapezoid, we first need to find an equation for the line. We see that the line passes through $(0, 16)$ and $(4, 0)$. Thus, the slope is

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - 16}{4 - 0} \\ &= -4. \end{aligned}$$

Remember that b is the y-intercept, which is 16 in this example. So an equation for the line is $v(t) = -4t + 16$.

Here, b_1 is always 16, b_2 corresponds to $v(t)$, and h is just t. This means that

$$\begin{aligned} s(t) &= \frac{1}{2} \cdot (b_1 + b_2) \cdot h \\ &= \frac{1}{2} \cdot (16 + v(t)) \cdot t \\ &= \frac{1}{2}(16 + (-4t + 16)) \cdot t \\ &= \frac{1}{2}(32 - 4t) \cdot t \\ &= 16t - 2t^2. \end{aligned}$$

Chapter 2

The First Derivative

2.1 The Derivative

Example 1

Please visit the link desmos page on Secant Lines for an interactive demonstration of the geometry of secant lines. The purpose of this demo is to see how slopes of secant lines approach the slope of the tangent line through a point.

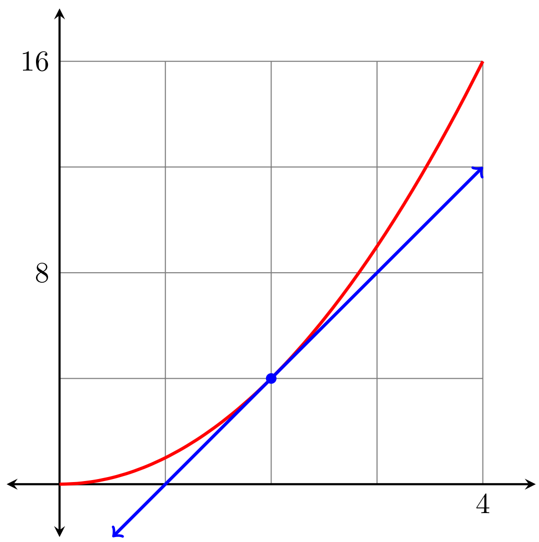


Figure 2.1: Graph of $y = x^2$ on the domain $[0, 4]$.

Why is this important? If you look at Figure 2.1, you can see the tangent line drawn through $(2, 4)$. Now to find the equation of this line, we either need two points, or a point and a slope. All we have is one point.

Using the geometry of secant lines, we will be able to calculate the slope of the tangent line. So let's look at this from an algebraic point of view. In the demonstration, we focused on the value $x = 3$, but now we'll look at the same geometry where x can be any value.

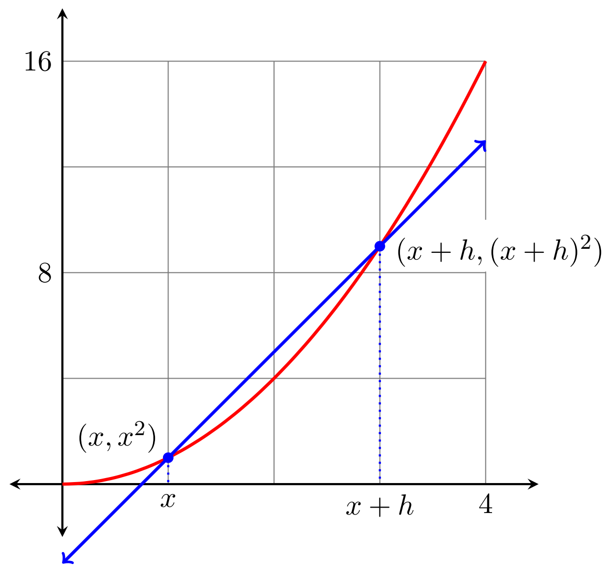


Figure 2.2: Graph of $y = x^2$ on the domain $[0, 4]$.

Begin with the point (x, x^2) on the graph of $y = x^2$. To find a secant line through this point, consider an x-value of $x+h$, with y-value $(x+h)^2$, so the other point on the graph is $(x+h, (x+h)^2)$. Draw the secant line between these two points, as shown in Figure 2.2.

Now use the slope formula to calculate the slope. Let the points be $(x_1, y_1) = (x, x^2)$ and $(x_2, y_2) = (x+h, (x+h)^2)$. Then

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(x+h)^2 - x^2}{(x+h) - x} \\ &= \frac{(x+h)^2 - x^2}{h} \end{aligned}$$

Remember, we want to let h get smaller and smaller until it's eventually 0. The problem is that if we try to substitute $h = 0$ at this point, we'll get $\frac{0}{0}$, which is undefined.

To solve this problem, we use the idea of a **limit** in calculus. The idea of a limit is *the fundamental* new concept in calculus which you likely haven't seen in precalculus. We're thinking along these lines: "If we look at the slopes of secant lines for small values of h, it looks like they're all approaching the same value. But we can't actually plug in $h = 0$, since we'll get $\frac{0}{0}$. So we need to take the limit of these slopes as h approaches 0."

There is a new calculus notation for this. It's

$$m = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}.$$

We read it as "m is the limit as $h \to 0$ of $(x+h)^2 - x^2$ over h."

Think of it this way. As we saw in the demo, the idea of a tangent line is very geometrical. But actually calculating the slopes of tangent lines takes a bit of algebra. The limit is the concept in calculus which brings together the geometrical and algebraic aspects of calculus.

Let's continue working to find the slope. The next step is to simplify our expression for m as much as possible and see where that leaves us.

$$\begin{aligned} m &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2hx + h^2}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= \lim_{h\to 0} (2x+h). \end{aligned}$$

That's as far as we can go. But do you notice what happened? *Now* we can plug in $h = 0$. That's because we canceled out the h from the denominator, so we're not dividing by 0 any more.

So we have

$$\begin{aligned} m &= \lim_{h\to 0} (2x+h) \\ &= 2x + 0 \\ &= 2x. \end{aligned}$$

But exactly what does the "$2x$" tell us? Look back at Figure 2.1. What is the equation of the tangent line at $x = 2$, which is the line shown there? Since the slope of a tangent line is $2x$, then the slope of this line has to be $2 \cdot 2 = 4$. This means that we are looking for the line passing through $(2, 4)$ with a slope of 4. Using the method of your choice, you get $y = 4x - 4$.

What have we done? Recall that the problem with finding an equation for a tangent line is that a tangent line is defined to be a line touching a curve at *one* point. We do not have a second point. But by using the concept of a *limit,* we can find the slope of the tangent line by taking a limit of the slopes of secant lines.

Let's now look at some common calculus notation. While we wrote $y = x^2$, very often we write $f(x) = x^2$. Depending on which notation is used (both are common), we would write $\dfrac{dy}{dx} = 2x$ or $f'(x) = 2x$, and call $2x$ the **derivative** of y, or the **derivative** of $f(x)$.

Rewriting our previous work using $f'(x)$, we would say that

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}.$$

But $f(x) = x^2$ and $f(x+h) = (x+h)^2$, so we could also say that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Written this way, we say that given the function $f(x)$, this is the **definition of the derivative** of $f(x)$. Deserves a double box.

Definition of the Derivative

If $f(x)$ is a function, we define the **derivative** of $f(x)$ to be $f'(x)$, which is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Example 2

Find the derivative of $f(x) = \sqrt{x}$.

Let's use our new definition to do this. We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}. \end{aligned}$$

How can we simplify this? The trick, which we saw earlier, is to rationalize the numerator. Let's see what happens.

$$\begin{aligned} \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\ &= \lim_{h\to 0} \frac{x + h + \sqrt{x+h}\cdot\sqrt{x} - \sqrt{x}\cdot\sqrt{x+h} - x}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}. \end{aligned}$$

Stop and look. What just happened here is because we rationalized the numerator, we got to a point where we could cancel the h in the denominator. And remember, since we want h to go to 0, we *have* to be able to cancel the h if we want to go any further.

Because when we cancel that h, *now* we can plug in $h = 0$.

$$\begin{aligned} \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} &= \frac{1}{\sqrt{x+0} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}. \end{aligned}$$

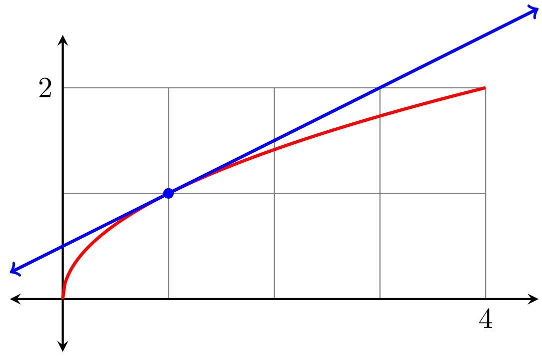


Figure 2.3: Graph of $y = \sqrt{x}$ on the domain $[0, 4]$.

Let's see how we would use the fact that $f'(x) = \dfrac{1}{2\sqrt{x}}$. A graph of $f(x)$ is shown in Figure 2.3, with a tangent line drawn at the point $(1, 1)$.

How would we get an equation of the tangent line? The slope is given by

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2},$$

and a point on the tangent line is $(1, 1)$. So an equation for the tangent line is $y = \dfrac{1}{2}x + \dfrac{1}{2}$.

Example 3

What is $f'(x)$ if $f(x) = \dfrac{1}{x}$? Again, we go back and use the definition of the derivative. Note that we multiply both top and bottom by $(x+h)x$ since that is the common denominator of the fractions in the numerator.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} \cdot \frac{(x+h)x}{(x+h)x} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h} \cdot (x+h)x - \dfrac{1}{x} \cdot (x+h)x}{h(x+h)x} \\
&= \lim_{h\to 0} \frac{x - (x+h)}{h(x+h)x} \\
&= \lim_{h\to 0} \frac{x - x - h}{h(x+h)x} \\
&= \lim_{h\to 0} \frac{-h}{h(x+h)x}
\end{aligned}$$

Now we can cancel out the h. When using the definition of the derivative, the h must *always* cancel out. *Always.* This is one way to know that you're on the right track. So

$$f'(x) = \lim_{h\to 0} \frac{-1}{(x+h)x}.$$

Because the h canceled, plugging in $h = 0$ is not a problem anymore. So

$$\begin{aligned} \lim_{h\to 0} \frac{-1}{(x+h)x} &= \frac{-1}{(x+0)x} \\ &= -\frac{1}{x^2}. \end{aligned}$$

One thing you probably noticed is that these problems have involved a *lot* of algebra. There is no way around this – you have to keep simplifying until you can get the h to cancel. Once you *do* get the h to cancel, you can be pretty sure you're on the right track.

Example 4

As mentioned, there is usually quite a bit of algebra in working with the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

So let's break it down into steps.

Let's start with $f(x) = x^2 - 3x$. How do we evaluate $f(x+h)$? We substitute $x + h$ everywhere we see an x. If you sometimes get stuck with this, here's something to try.

1. First, rewrite the function with boxes.

$$f\left(\boxed{}\right) = \left(\boxed{}\right)^2 - 3\left(\boxed{}\right).$$

2. Next, put $x + h$ in each empty box.

$$f\left(\boxed{x+h}\right) = \left(\boxed{x+h}\right)^2 - 3\left(\boxed{x+h}\right).$$

3. We don't need the boxes any more.

$$f(x+h) = (x+h)^2 - 3(x+h).$$

4. Expand. Be careful when distributing the minus sign.

$$f(x+h) = x^2 + 2xh + h^2 - 3x - 3h.$$

5. Now substitute into the limit definition and simplify until the h cancels. Again, watch the minus signs. Note that for the h to cancel, *every* term in the numerator that does *not* contain h should cancel.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-(x^2-3x)}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x}{h} \\
&= \lim_{h\to 0} \frac{2xh+h^2-3h}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h-3)}{h} \\
&= \lim_{h\to 0} (2x+h-3) \\
&= 2x-3.
\end{aligned}$$

Assessment Expectations: You do *not* have to find the derivative using boxes. But I have found that many students start off with an incorrect expression for $f(x+h)$. If your first step isn't correct, it's going to be difficult to get the h to cancel. Use it only if it helps you.

Homework

1. Using the definition of the derivative, find $f'(x)$ if $f(x) = -3x$.
2. (a) Using the definition of the derivative, find $f'(x)$ if $f(x) = x - 2x^2$.
   (b) Find the equation of the tangent line at $x = 1$. Graph both $f(x)$ and the tangent line on desmos to visually verify that you have the correct tangent line.
3. (a) Using the definition of the derivative, find $f'(x)$ if $f(x) = \sqrt{x+2}$.
   (b) Find the equation of the tangent line at $x = 2$. Graph both $f(x)$ and the tangent line on desmos to visually verify that you have the correct tangent line.
4. (a) Using the definition of the derivative, find $f'(x)$ if $f(x) = \dfrac{1}{x-1}$.
   (b) Find the equation of the tangent line at $x = 3$. Graph both $f(x)$ and the tangent line on desmos to visually verify that you have the correct tangent line.

Solutions

1.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{-3(x+h)-(-3x)}{h} \\
&= \lim_{h\to 0} \frac{-3x-3h+3x}{h} \\
&= \lim_{h\to 0} \frac{-3h}{h} \\
&= \lim_{h\to 0} (-3) \\
&= -3.
\end{aligned}
$$

2. (a)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{x+h-2(x+h)^2-(x-2x^2)}{h} \\
&= \lim_{h\to 0} \frac{x+h-2x^2-4xh-2h^2-x+2x^2}{h} \\
&= \lim_{h\to 0} \frac{h-4xh-2h^2}{h} \\
&= \lim_{h\to 0} \frac{h(1-4x-2h)}{h} \\
&= \lim_{h\to 0} (1-4x-2h) \\
&= 1-4x-2(0) \\
&= 1-4x.
\end{aligned}
$$

(b) $f(1) = -1$ and $f'(1) = 1-4(1) = -3$, so the slope of the line is -3 and the line passes through $(1,-1)$. This results in the line $y = -3x+2$.

3. (a)

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{x+2+h}-\sqrt{x+2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+2}-\sqrt{x+2}}{h} \cdot \frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}} \\
&= \lim_{h\to 0} \frac{x+h+2+\sqrt{x+h+2}\cdot\sqrt{x+2}-\sqrt{x+2}\cdot\sqrt{x+h+2}-x-2}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2}+\sqrt{x+2})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2}+\sqrt{x+2}} \\
&= \frac{1}{\sqrt{x+0+2}+\sqrt{x+2}} \\
&= \frac{1}{\sqrt{x+2}+\sqrt{x+2}} \\
&= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

(b) $f(2) = 2$ and $f'(2) = \dfrac{1}{2\sqrt{2+2}} = \dfrac{1}{4}$, and so we are looking for a line with slope $\dfrac{1}{4}$ which passes through the point $(2, 2)$. This results in $y = \dfrac{1}{4}x + \dfrac{3}{2}$.

4. (a)

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h} \cdot \frac{(x+h-1)(x-1)}{(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{\dfrac{1}{x+h-1} \cdot (x+h-1)(x-1) - \dfrac{1}{(x-1)} \cdot (x+h-1)(x-1)}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{x-1-(x+h-1)}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{x-1-x-h+1}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{-h}{h(x+h-1)(x-1)} \\
&= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} \\
&= \frac{-1}{(x+0-1)(x-1)} \\
&= -\frac{1}{(x-1)^2}
\end{aligned}
$$

(b) $f(3) = \dfrac{1}{2}$ and $f'(3) = -\dfrac{1}{(3-1)^2} = -\dfrac{1}{4}$, so we are looking for a line with slope $-\dfrac{1}{4}$ which passes through the point $\left(3, \dfrac{1}{2}\right)$. This line is $y = -\dfrac{1}{4}x + \dfrac{5}{4}$.

2.2 The Derivative of $y = \sin(x)$.

Let's look at the derivatives of a few common functions. We'll start with $f(x) = \sin(x)$; two full periods are graphed in Figure 2.4. What is the slope of the tangent line at $x = 0$? In other words, what is $f'(0)$?

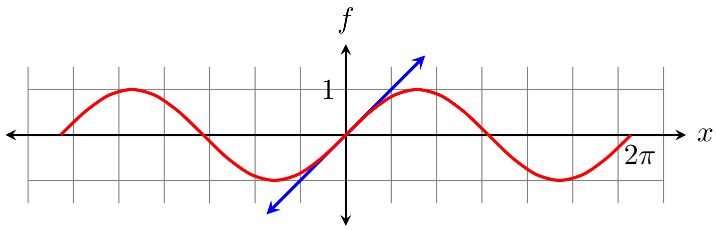


Figure 2.4: Graph of $f(x) = \sin(x)$ with tangent line at $x = 0$.

We will use the following definition of the derivative.

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

We first start with $x = 0$, write out the definition substituting in 0 for x, and then simplify a little bit.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{\sin(h) - \sin(0)}{h} && \text{since } f(x) = \sin(x) \\ &= \lim_{h \to 0} \frac{\sin(h)}{h} && \text{since } \sin(0) = 0. \end{aligned}$$

Up until now, we were able to use algebra to make the "h" cancel out so we could just substitute $h = 0$. But it is not possible to do that here. So how do we proceed?

There are two other ways we can look at limits: numerically and graphically. We'll start with numerically. Since we are looking at a limit at $h \to 0$, you can use your calculator to look at the quotient $\dfrac{\sin(h)}{h}$ for values of h closer and closer to 0.

I set my calculator to radian mode (important!) and rounded to six decimal places. As h gets closer to 0 from the left and right, it looks like the quotient $\dfrac{\sin(h)}{h}$ gets closer and closer to 1. Using limit notation, we would write

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1.$$

h	$\sin(h)/h$
-0.1	0.998334
-0.01	0.999983
-0.001	1.000000
-0.0001	1.000000
0.1	0.998334
0.01	0.999983
0.001	1.000000
0.0001	1.000000

Table 2.1: Approximating $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h}$.

It is worth noting that if your calculator were in degree mode, it would look like this limit is approximately 0.017453. Units of radians make trigonometry much easier (as far as calculus is concerned). This is very similar to choosing appropriate units in science. The metric system is far better suited to science than inches, ounces, etc.

Another way to guess $\lim\limits_{h\to 0} \dfrac{\sin(h)}{h}$ is to look at the ratio $\dfrac{\sin(h)}{h}$ as a function itself, as in Figure 2.5.

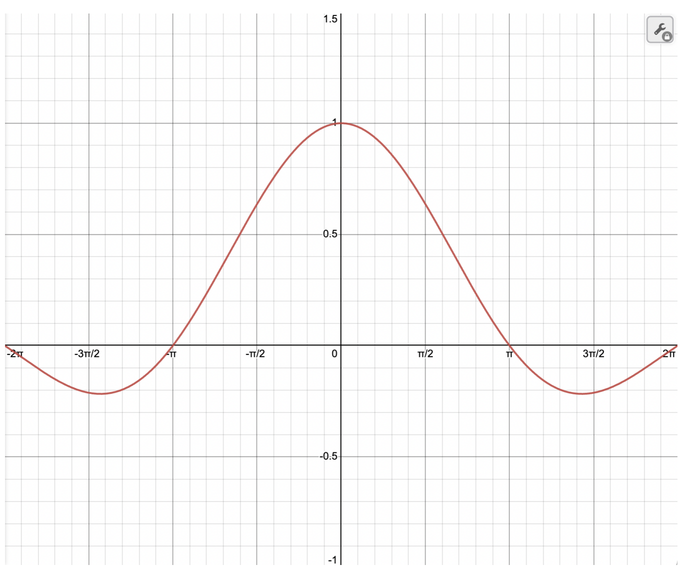


Figure 2.5: Graph of $g(h) = \dfrac{\sin(h)}{h}$.

I made this graph with desmos and took a screen shot. It looks like it crosses the y-axis at 1 (just like the limit). It's important to say "looks like" since you can't actually evaluate

$$g(0) = \frac{\sin(0)}{0} = \frac{0}{0}.$$

But most graphing programs are able to "fill in the hole" at $x = 0$ to get a smooth curve.

There is also a more complicated mathematical proof using geometry and trigonometry, but it's more than we need. For the functions we'll be looking at, if looking at a limit numerically and graphically gives the same result, then you can be sure you've found the right limit.

To recap, we found that

$$\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1.}$$

But this was the limit we needed to evaluate to find the slope of the tangent line at $x = 0$, and so the slope of this line is 1.

We started just with looking at $x = 0$ since we needed to see different ways to evaluate limits. So with $f(x) = \sin(x)$, let's find $f'(x)$ for *every* x. To do this, we'll need an identity from trigonometry:

$$\boxed{\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b).}$$

Now let's start with the definition.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} && \text{since } f(x) = \sin(x) \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} && \text{using the identity}
\end{aligned}$$

This looks a bit more complicated than other limits we've seen. Let's take a few steps to rearrange terms.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \left[\sin(x) \cdot \frac{\cos(h)}{h} + \cos(x) \cdot \frac{\sin(h)}{h} - \frac{\sin(x)}{h}\right] && \text{splitting apart} \\
&= \lim_{h\to 0} \left[\sin(x) \left(\frac{\cos(h) - 1}{h}\right) + \cos(x) \cdot \frac{\sin(h)}{h}\right] && \text{combining } \sin(x) \text{ terms}
\end{aligned}$$

Now since h goes to 0, x does not change in this limit. So we can factor out terms *only* containing x. If the limit involves $h \to a$, you can *never* factor out any expression containing h from the limit.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left[\sin(x)\left(\frac{\cos(h)-1}{h}\right)+\cos(x)\cdot\frac{\sin(h)}{h}\right] \\ &= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \end{aligned}$$

The first limit might not look familiar, but the second one does – we started off by finding this exact limit: it is 1.

What about the first limit? Since we already worked one limit like this in detail, we won't do another one. But when you look at this limit numerically and graphically, you see that:

$$\boxed{\lim_{h\to 0}\frac{\cos(h)-1}{h}=0.}$$

How can we use this? Basically, we rewrote the quotient $\dfrac{f(x+h)-f(x)}{h}$ in such a way that is involves limits we can derive numerically and graphically. So we just substitute in the values of these limits.

$$\begin{aligned} f'(x) &= \sin(x)\cdot\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \\ &= \sin(x)\cdot 0+\cos(x)\cdot 1 \\ &= \cos(x). \end{aligned}$$

Done! So when $f(x)=\sin(x)$, then $f'(x)=\cos(x)$. So $\cos(x)$ is the derivative of $\sin(x)$, which we often write

$$\boxed{\frac{d}{dx}\sin(x)=\cos(x).}$$

Exercises

1. Show numerically and graphically that
$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0.$$

2. By following a similar sequence of steps as for $\sin(x)$, but using a different trigonometric identity, show that
$$\frac{d}{dx} \cos(x) = -\sin(x).$$

3. Let $f(x) = \sqrt{x+1}$, so that $f'(x) = \dfrac{1}{2\sqrt{x+1}}$.
   (a) Graph this function on desmos.
   (b) Where is $f'(x)$ defined?
   (c) What is $f'(3)$?
   (d) Find the equation of the tangent line at $x = 3$.

4. Let $f(x) = x^3 - 3x$.
   (a) Graph this function on desmos.
   (b) Using the definition of the derivative, show that $f'(x) = 3x^2 - 3$.
   (c) Where is $f'(x)$ defined?
   (d) What is $f'(2)$?
   (e) Find the equation of the tangent line at $x = 2$.

Solutions

1. Create a chart like we did for $\dfrac{\sin(h)}{h}$ and observe the numbers keep getting closer to 0 as h gets closer to 0. Note: if your numbers are not matching, be sure your calculator is in radian mode.

h	$(\cos(h)-1)/h$
-0.1	0.049958
-0.01	0.005000
-0.001	0.000500
-0.0001	0.000050
0.1	0.049958
0.01	0.005000
0.001	0.000500
0.0001	0.000050

You can check this graphically using desmos or a graphing calculator.

2. The identity we need is

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Now let's use the definition of the derivative with $f(x) = \cos(x)$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h} && \text{since } f(x)=\cos(x) \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} && \text{using the identity}
\end{aligned}$$

Now rearrange terms.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} \\
&= \lim_{h\to 0} \left[\cos(x)\cdot\frac{\cos(h)}{h} - \sin(x)\cdot\frac{\sin(h)}{h} - \cos(x)\cdot\frac{1}{h}\right] && \text{splitting apart} \\
&= \lim_{h\to 0} \left[\cos(x)\left(\frac{\cos(h)-1}{h}\right) - \sin(x)\cdot\frac{\sin(h)}{h}\right] && \text{combining } \cos(x) \text{ terms}
\end{aligned}$$

Recall the following limits.

$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0, \qquad \lim_{h\to 0}\frac{\sin(h)}{h} = 1.$$

Substituting in, we get

$$\begin{aligned} f'(x) &= \cos(x) \cdot 0 - \sin(x) \cdot 1 \\ &= -\sin(x) \end{aligned}$$

3. (a) Do this online.
   (b) $f'(x)$ is defined when $x + 1 > 0$, since you can't have 0 in the denominator or a negative number inside a square root. So $x > -1$, or $(-1, \infty)$ using interval notation.
   (c)
   $$f'(3) = \frac{1}{2\sqrt{3+1}} = \frac{1}{4}.$$
   (d) We know the slope is $\frac{1}{4}$ from (c). $f(3) = \sqrt{3+1} = 2$, so a point on the line is $(3, 2)$.
   $$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= \frac{1}{4}(x - 3) \\ y - 2 &= \frac{1}{4}x - \frac{3}{4} \\ y &= \frac{1}{4}x + \frac{5}{4} \end{aligned}$$

   (a) Do this online.
   (b)
   $$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - 3(x+h) - (x^3 - 3x)}{h} \\ &= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{3x} - 3h - \cancel{x^3} + \cancel{3x}}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\ &= \lim_{h\to 0} \left(\frac{3x^2h}{h} + \frac{3xh^2}{h} + \frac{h^3}{h} - \frac{3h}{h} \right) \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 3) \\ &= 3x^2 - 3. \end{aligned}$$
   (c) Polynomials are defined for *all* x. In interval notation, $f(x)$ is defined on $(-\infty, \infty)$.
   (d)
   $$f'(2) = 3(2^2) - 3 = 9.$$
   (e) We know the slope is 9 from (d). Since $f(2) = 2$, we use the point $(2, 2)$.
   $$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 9(x - 2) \\ y - 2 &= 9x - 18 \\ y &= 9x - 16 \end{aligned}$$

2.3 The Geometry of Derivatives

We just learned how to find a derivative using the geometric definition derived from looking at secant lines. The process of finding a derivative *algebraically* is sometimes rather tedious. Here, we'll look at the *geometrical* meaning of the derivative. Because we want to emphasize the important concepts, we'll look at a basic function, $f(x) = x^2$, shown below, with it's derivative, $f'(x) = 2x$.

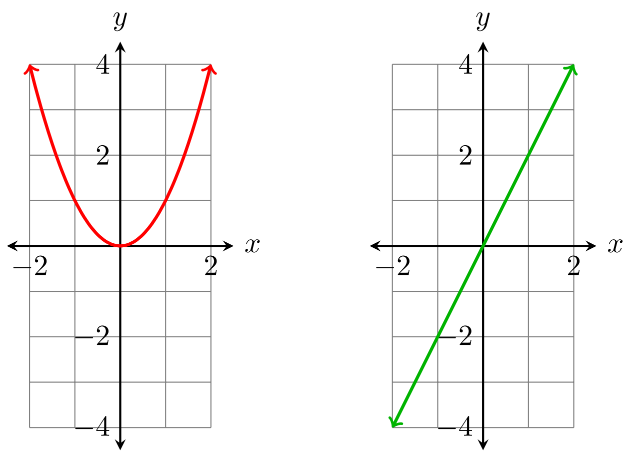


Figure 2.6: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right).

Let's look at exactly what knowledge we can gain by knowing the derivative. First, we can find the slope of a tangent at any given point. So, since $f(1) = 1$, we know that the tangent line goes through $(1, 1)$. And since $f'(x) = 2x$, this line has a slope of $f'(1) = 2 \cdot 1 = 2$, shown below. Note the corresponding point on the derivative graph, $(1, 2)$.

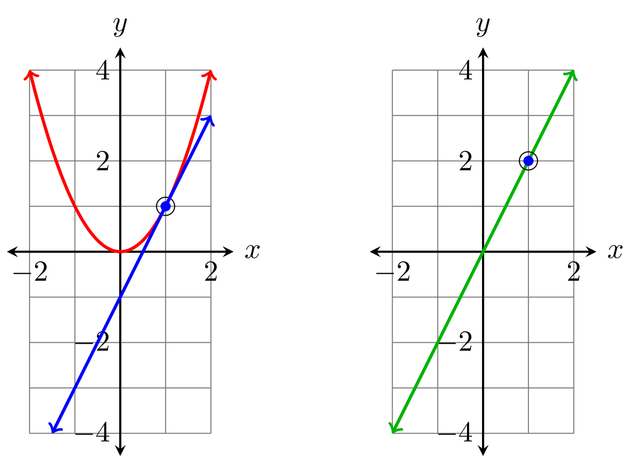


Figure 2.7: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right).

We now have enough information to work out an equation for the tangent line at $x = 1$, since we know the slope $m = 2$ and a point $(1, 1)$ on the line. For reference, we recall that a line with slope m which passes through the point (x_1, y_1) can be described by the following equation:

$$\boxed{y - y_1 = m(x - x_1).}$$

Substituting in our values: $m = 2$, $x_1 = 1$, and $y_1 = 1$, we get

$$y - 1 = 2(x - 1),$$

which simplifies to $y = 2x - 1$. In Figure 2.7, you can observe that the slope is 2 and the y-intercept is -1. So we can use the derivative to find an equation of the tangent line at a specific point.

What else does the derivative tell us? Let's look now at the case when $x > 0$; we look at the specific case $x = 0.5$ in the graphs below.

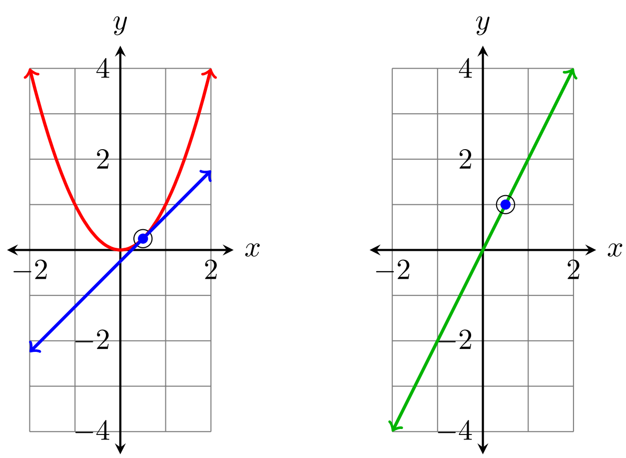


Figure 2.8: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right); $x = 0.5$.

Notice that the graph is *increasing* when $x > 0$, and so the tangent line has a *positive* slope. We can see this by looking at the graph. But in addition to this, we have, when $x > 0$,

$$\begin{aligned} f'(x) &= 2x \\ &> 2 \cdot 0 && \text{since } x > 0 \\ &= 0. \end{aligned}$$

We can summarize this as follows:

If $f'(x) > 0$ for some value of x, then the function $f(x)$ is increasing at that value of x.

Now let's look at the case when $x < 0$; the case when $x = -1.5$ is graphed in Figure 2.9.

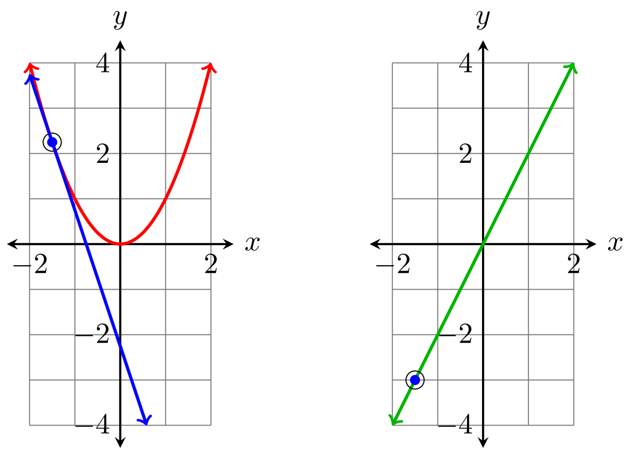


Figure 2.9: Graph of $f(x) = x^2$ (left) with derivative $f'(x) = 2x$ (right); $x = -1.5$.

In this case, the graph is *decreasing* when $x < 0$, and so the tangent line has a *negative* slope. We can see this by looking at the graph. But we can also see this algebraically; when $x < 0$,

$$\begin{aligned} f'(x) &= 2x \\ &< 2 \cdot 0 && \text{since } x < 0 \\ &= 0. \end{aligned}$$

We can summarize this case as follows:

If $f'(x) < 0$ for some value of x, then the function $f(x)$ is decreasing at that value of x.

Let's summarize what we've observed so far.

$f'(x)$	Where	What happens
$f'(x) < 0$	$(-\infty, 0)$	$f(x)$ is decreasing
$f'(x) > 0$	$(0, \infty)$	$f(x)$ is increasing

So we can understand some features of the graph of a function by looking at its derivative. It is *always* true that if $f'(x) < 0$, then $f(x)$ is decreasing, and if $f'(x) > 0$, then $f(x)$ is increasing. But the case when $f'(x) = 0$ is a little trickier. We'll be looking at this case in detail later.

Exercises

1. Using the definition of the derivative, find $f'(x)$ if $f(x) = 3$. Write a short sentence interpreting this geometrically.

2. Using the definition of the derivative, show that if $f(x) = ax$, then $f'(x) = a$. In this example, a is just a number, like 3. Write a short sentence interpreting this geometrically.

3. Let $f(x) = \cos(x)$.

   (a) Graph this function on desmos. Use a domain of $[0, 2\pi]$ (you can use the wrench icon in the upper right and type "pi" for π).

   (b) What is $f'\left(\frac{\pi}{2}\right)$?

   (c) Find the equation of the tangent line at $x = \frac{\pi}{2}$.

   (d) Where is $f'(x) = 0$? Remember, the domain is $[0, 2\pi]$. By inspecting the graph, decide if there is a minimum, maximum, or inflection point at these values.

   (e) Where is the function increasing?

   (f) Where is the function decreasing?

4. Let $f(x)$ be a function – you don't know exactly what $f(x)$ is, but you are given that $f'(x) = x^2(x-2)^2$. The function is defined on all real numbers.

   (a) Where is this function increasing?

   (b) Where is this function decreasing?

   (c) When is $f'(x) = 0$? Based on what you found in (a) and (b), decide if $f(x)$ has a minimum, maximum, or inflection point at these values.

   (d) You are given that $f(3) = 12$. Find an equation of the tangent line at $x = 3$.

Solutions

1.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{3-3}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0 \end{aligned}$$

Since $y = 3$ is a horizontal line, this means that its slope is 0.

2.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{a(x+h)-ax}{h} \\ &= \lim_{h\to 0} \frac{ax+ah-ax}{h} \\ &= \lim_{h\to 0} \frac{ah}{h} \\ &= \lim_{h\to 0} a \\ &= a. \end{aligned}$$

This confirms that the line $y = ax$ is a line with slope a.

3. (a) Do this online.

(b) $f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$

(c) We know the slope is -1 from (b). Since $f\left(\frac{\pi}{2}\right) = 0$, we use the point $\left(\frac{\pi}{2}, 0\right)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -1\left(x - \frac{\pi}{2}\right) \\ y &= -x + \frac{\pi}{2} \end{aligned}$$

(d) $f'(x) = -\sin(x) = 0$ exactly when $x = 0, \pi, 2\pi$ (given our knowledge of the unit circle and the fact that the domain is $[0, 2\pi]$). Looking at the graph, we see a local maximum at $x = 0$ and $x = 2\pi$, and a local minimum at $x = \pi$.

(e) The function is increasing wherever we have $f'(x) > 0$. Looking at the graph of $y = -\sin(x)$ on desmos, we observe that $f'(x) = -\sin(x)$ is positive on the interval $(\pi, 2\pi)$. By visually inspecting the graph of $f(x) = \cos(x)$, we observe that $f(x)$ is increasing on this interval.

(f) The function is decreasing wherever we have $f'(x) < 0$. Looking at the graph of $y = -\sin(x)$ on desmos, we observe that $f'(x) = -\sin(x)$ is negative on the interval $(0, \pi)$. By visually inspecting the graph of $f(x) = \cos(x)$, we observe that $f(x)$ is decreasing on this interval.

4. In this problem, you are not given the graph of the function, but you should still be able to answer the following questions.

   (a) $f'(x) = x^2(x-2)^2$, but x^2 and $(x-2)^2$ are both positive. So $f'(x)$ is always positive, therefore $f(x)$ is always increasing. In interval notation, this would be $(-\infty, \infty)$.

   (b) Based on the answer to (a), $f(x)$ is never decreasing.

   (c) Since $f'(x)$ is in factored form, the zeros are easy to find: $x = 0$ and $x = 2$. Now if there were a minimum at $x = 0$, we would go from decreasing to increasing, which is impossible since $f(x)$ is never decreasing. Likewise, if there were a maximum, we would go from increasing to decreasing, again impossible. So there must be inflection points at these two values of x.

   (d) Since $f'(3) = 3^2 \cdot (3-2)^2 = 9$, the slope of the tangent line is 9. Since $f(3) = 12$, we know that $(3, 12)$ is a point on the tangent line. We can use these to get an equation of the line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 12 &= 9(x - 3) \\ y - 12 &= 9x - 27 \\ y &= 9x - 15 \end{aligned}$$

Chapter 3

Using Rules of Differentiation

3.1 Rules of Differentiation

We're finished using the definition of the derivative for a while. Now we want to learn how to use some rules which will make differentiating more complex functions possible.

Example 1

We have seen that $\frac{d}{dx}x = 1$, $\frac{d}{dx}x^2 = 2x$. Using the definition of the derivative, we see that

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{h\to 0}\frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0}\frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \\
&= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0}\left(\frac{3x^2h}{h} + \frac{3xh^2}{h} + \frac{h^3}{h}\right) \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\
&= 3x^2.
\end{aligned}$$

Note that dividing through by h is an alternative to factoring the h out. Both methods will work.

This pattern continues, and we have the Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

This works even if the exponent is not a positive integer. We've seen this twice before. Using the Power Rule, we have

$$\begin{aligned}
\frac{d}{dx}\frac{1}{x} &= \frac{d}{dx}x^{-1} \\
&= -1x^{-1-1} \\
&= -x^{-2} \\
&= -\frac{1}{x^2}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}\sqrt{x} &= \frac{d}{dx}x^{1/2} \\
&= \frac{1}{2}x^{1/2-1} \\
&= \frac{1}{2}x^{-1/2} \\
&= \frac{1}{2\sqrt{x}},
\end{aligned}$$

both found before using the definition of the derivative. Now we have the Power Rule, and so can use this instead of the definition.

Example 2

We have seen that $\dfrac{d}{dx}x^2 = 2x$ and $\dfrac{d}{dx}\sin(x) = \cos(x)$. Adding or subtracting functions, or multiplying functions by a number, do *not* affect the algebra needed in the definition of a derivative. So

$$\begin{aligned}
\frac{d}{dx}(x^2 + \sin(x)) &= 2x + \cos(x) \\
\frac{d}{dx}(\sin(x) - x^2) &= \cos(x) - 2x \\
\frac{d}{dx}5x^2 &= 5(2x) = 10x \\
\frac{d}{dx}(-3\sin(x)) &= -3\cos(x)
\end{aligned}$$

Example 3

However, multiplying and dividing functions *does* affect the algebra when using the definition of the derivative. So you cannot just multiply or divide derivatives. Let's look at multiplying functions first. Here, we use the Product Rule:

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

We won't prove this here, but let's look at an example to see how to use it. We'll find $\dfrac{d}{dx}x\cos(x)$. Here, $f(x) = x$ and $g(x) = \cos(x)$, so that

$$\begin{aligned}
f(x) &= x & \qquad f'(x) &= 1 \\
g(x) &= \cos(x) & \qquad g'(x) &= -\sin(x)
\end{aligned}$$

Now substitute in the Product Rule.

$$\begin{aligned}
\frac{d}{dx}x\cos(x) &= f(x)g'(x) + g(x)f'(x) \\
&= x(-\sin(x)) + \cos(x) \cdot 1 \\
&= -x\sin(x) + \cos(x)
\end{aligned}$$

This is much easier than going back and using the definition of the derivative.

Example 4

When dividing functions, we use the Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

This looks a bit more complicated, but as long as you substitute carefully, you'll be fine. We'll use it to find $\frac{d}{dx}\frac{\sin(x)}{x^2}$. Here are the substitutions:

$$\begin{aligned} f(x) &= \sin(x) & f'(x) &= \cos(x) \\ g(x) &= x^2 & g'(x) &= 2x \end{aligned}$$

Now substitute into the Quotient Rule:

$$\begin{aligned} \frac{d}{dx}\frac{\sin(x)}{x^2} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{x^2 \cdot \cos(x) - \sin(x)(2x)}{(x^2)^2} \\ &= \frac{x^2\cos(x) - 2x\sin(x)}{x^4} \\ &= \frac{x(x\cos(x) - 2\sin(x))}{x^4} \\ &= \frac{x\cos(x) - 2\sin(x)}{x^3} \end{aligned}$$

Notice that the x cancels. Be sure to make any simple cancellations when possible.

Example 5

Let's briefly review function composition. Suppose $f(x) = \cos(x)$ and $g(x) = x^3$. Then

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= \cos(g(x)) \\ &= \cos(x^3). \end{aligned}$$

For taking derivatives using the Chain Rule (done in the next section), we'll need to do this in *reverse.* As an example, if $h(x) = (3x+2)^4$, find $f(x)$ and $g(x)$ such $h(x) = f(g(x))$.

To think about this, notice that $g(x)$ is the function you evaluate *first,* and $f(x)$ is the function you evaluate *last.* Think about how you would evaluate $h(x)$ using your calculator. If you have to find $h(x)$, the first thing you'd do is evaluate $3 \cdot 5 + 2 = 17$, and the last thing you'd do is take 17^4. So $g(x) = 3x + 2$ and $f(x) = x^4$.

Summary of Rules of Differentiation

Power Rule:	$\dfrac{d}{dx}(x^n) = nx^{n-1}$
Sum Rule:	$\dfrac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
Difference Rule:	$\dfrac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
Constant Multiple Rule:	$\dfrac{d}{dx}(cf(x)) = cf'(x)$
Product Rule:	$\dfrac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$
Quotient Rule:	$\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$
Chain Rule:	$\dfrac{d}{dx}(f \circ g)(x) = f'(g(x))g'(x)$

Exercises

1. Find the deriviatives of the following functions using the appropriate rule.

   (a) $h(x) = 3x^5 - x^3$

   (b) $h(x) = 2\cos(x) - x^2$

   (c) $h(x) = x^3 \sin(x)$

   (d) $h(x) = \dfrac{x^2}{\cos(x)}$

2. Review Paul's Online Notes on function composition, if necessary. For each of the following functions, find $f(x)$ and $g(x)$ so that $h = f \circ g$.

   (a) $h(x) = (2x-1)^5$

   (b) $h(x) = \sin^2(x)$

   (c) $h(x) = \dfrac{1}{x^3 + x}$

   (d) $h(x) = \cos(2x+1)$

Solutions

1. (a)

$$\begin{aligned} h'(x) &= \frac{d}{dx}(3x^5 - x^3) \\ &= 3(5x^4) - 3x^2 \\ &= 15x^4 - 3x^2 \end{aligned}$$

(b)

$$\begin{aligned} h'(x) &= \frac{d}{dx}(2\cos(x) - x^2) \\ &= 2(-\sin(x)) - 2x \\ &= -2\sin(x) - 2x \end{aligned}$$

(c) Use the following substitutions in the Product Rule.

$$f(x) = x^3 \qquad f'(x) = 3x^2$$
$$g(x) = \sin(x) \qquad g'(x) = \cos(x)$$

Then

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^3\cos(x) + \sin(x) \cdot 3x^2 \\ &= x^3\cos(x) + 3x^2\sin(x). \end{aligned}$$

(d) Use the following substitutions in the Quotient Rule.

$$f(x) = x^2 \qquad f'(x) = 2x$$
$$g(x) = \cos(x) \qquad g'(x) = -\sin(x)$$

Then

$$\begin{aligned} h'(x) &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{\cos(x) \cdot 2x - x^2(-\sin(x))}{(\cos(x))^2} \\ &= \frac{2x\cos(x) + x^2\sin(x)}{\cos^2(x)} \end{aligned}$$

2. (a) $f(x) = x^5$, $g(x) = 2x - 1$.
   (b) $f(x) = x^2$, $g(x) = \sin(x)$.
   (c) $f(x) = \dfrac{1}{x}$, $g(x) = x^3 + x$.
   (d) $f(x) = \cos(x)$, $g(x) = 2x + 1$.

3.2 Using Differentiation Rules

When you are given a function $f(x)$ and need to take the derivative, which rule(s) should you use? It's not always obvious.

Example 1

Let $f(x) = \dfrac{x^4 - 3x^2 + 5x}{x}$. Since $f(x)$ is a fraction, your first thought might be to use the quotient rule. This isn't wrong, but it's way too much work. In this case, it's easier to divide out the x first.

$$\begin{aligned} f(x) &= \frac{x^4 - 3x^2 + 5x}{x} \\ &= \frac{x^4}{x} - \frac{3x^2}{x} + \frac{5x}{x} \\ &= x^3 - 3x + 5 \end{aligned}$$

Now, it's easy to find $f'(x)$ using the power rule: $f'(x) = 3x^2 - 3$.

Example 2

Let $f(x) = \dfrac{\cos(x)}{x^{-2}}$. Again, it's tempting to use the quotient rule. But recall that

$$\frac{1}{x^{-2}} = x^2.$$

So it's easier to write $f(x) = x^2 \cos(x)$ and use the product rule.

$$\begin{aligned} f(x) &= x^2 \cos(x) \\ f'(x) &= (x^2)(-\sin(x)) + \cos(x)(2x) \\ &= -x^2 \sin(x) + 2x \cos(x). \end{aligned}$$

Example 3

Let $f(x) = (x^2 + 1)(x - 3)$. Yes, you can use the product rule here. But in this case, it's simpler to FOIL out $f(x)$ and then just use the power rule.

$$\begin{aligned} f(x) &= (x^2 + 1)(x - 3) \\ &= x^3 - 3x^2 + x - 3 \\ f'(x) &= 3x^2 - 6x + 1 \end{aligned}$$

The common theme here is that we rewrote each function so that an easier differentiation rule can be used. There's no "magic formula" for how to do this, you just have to practice. But before jumping into a problem, it's always a good idea to take a moment to see if the function can be rewritten to make it easier to differentiate.

Example 4

Usually, when you see a function inside of another function, you need to use the chain rule. Occasionally, there may be a different way. Suppose $h(x) = (x^3 + 1)^2$. You might try the chain rule with $f(x) = x^2$ and $g(x) = x^3 + 1$. Then $f'(x) = 2x$ and $g'(x) = 3x^2$, so that

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 2(g(x)) \cdot 3x^2 \\ &= 2(x^3 + 1)(3x^2) \\ &= 6x^2(x^3 + 1). \end{aligned}$$

But it also possible to FOIL out $f(x)$ first. Here's what you get.

$$\begin{aligned} h(x) &= (x^3 + 1)^2 \\ &= x^6 + 2x^3 + 1 \\ h'(x) &= 6x^5 + 6x^2 \end{aligned}$$

Since $6x^2(x^3+1) = 6x^5+6x^2$, both methods give the same answer. One way isn't necessarily easier than the other, so either way you choose to do it is OK.

Example 5

Let $f(x) = \dfrac{5}{x^6}$. This is a fraction, so you might be tempted to use the quotient rule. But it's easier to use rules of exponents to rewrite $f(x) = 5x^{-6}$. You cannot just use the power rule on the denominator; the *entire* function must be of the form ax^n for some n.

$$\begin{aligned} f(x) &= 5x^{-6} \\ f'(x) &= 5 \cdot (-6)x^{-6-1} \\ &= -30x^{-7} \end{aligned}$$

Note that the exponent must be "-7," not "-5," since we have to subtract 1 from the exponent.

Find the derivatives of the following functions. Some problems will be easier if you rewrite them first, so take a moment to look before you leap.

1. $h(x) = \dfrac{2x}{3x^4}$
2. $h(x) = \dfrac{x^2+1}{x}$
3. $h(x) = \dfrac{x}{x^2+1}$
4. $h(x) = (5x-3)^{10}$
5. $h(x) = (5x-3)^{-10}$
6. $h(x) = x^3\sqrt{x}$
7. $h(x) = \sin(x)\sqrt{x}$
8. $h(x) = \dfrac{ax+b}{ax-b}$ Hint: Treat a and b like numbers, so $\dfrac{d}{dx}a = \dfrac{d}{dx}b = 0$.
9. $h(x) = \sin(x)\cos(x)$
10. $h(x) = \sqrt{3x-5}$
11. $h(x) = \cos(x^3)$
12. $h(x) = \cos^3(x)$
13. $h(x) = \tan(x)$ (Hint: Use the quotient rule.)
14. $h(x) = \dfrac{\sin(x)}{x^{-3}}$
15. For each of the following, simplify/rewrite if possible, and state which rule you would use to take the derivative. Do not actually take the derivative. This problem is for helping you to decide which rule to use.
    (a) $\sin(6x^2+1)$
    (b) $\dfrac{3}{x^7}$
    (c) $\dfrac{\cos(x)}{x^{-2}}$
    (d) $x^2\sqrt{x}$
    (e) $\dfrac{x^3-3x^2}{x^2}$
    (f) $\dfrac{x+1}{x-1}$
    (g) $(x^2+1)^8$
    (h) $\dfrac{3}{x^{-4}}$

Solutions

1. First, rewrite.

$$\begin{aligned}\frac{2x}{3x^4} &= \frac{2}{3x^3}\\ &= \frac{2}{3}x^{-3}\end{aligned}$$

   Then use the Power Rule.

$$\begin{aligned}\frac{d}{dx}\frac{2}{3}x^{-3} &= \frac{2}{3}\cdot(-3)x^{-3-1}\\ &= -2x^{-4}\\ &= -\frac{2}{x^4}\end{aligned}$$

2. First rewrite.

$$\begin{aligned}\frac{x^2+1}{x} &= \frac{x^2}{x}+\frac{1}{x}\\ &= x+x^{-1}\end{aligned}$$

   Then use the Power Rule.

$$\begin{aligned}\frac{d}{dx}(x^1+x^{-1}) &= 1\cdot x^{1-1} - 1\cdot x^{-1-1}\\ &= 1-x^{-2}\\ &= 1-\frac{1}{x^2}\end{aligned}$$

3. Here, we need to use the Quotient Rule.

$$\begin{aligned}f(x) &= x & f'(x) &= 1\\ g(x) &= x^2+1 & g'(x) &= 2x\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\frac{x}{x^2+1} &= \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}\\ &= \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2}\\ &= \frac{x^2+1-2x^2}{(x^2+1)^2}\\ &= \frac{1-x^2}{(x^2+1)^2}\end{aligned}$$

   Note that we do *not* expand the denominator when using the Quotient Rule.

4. Use the Chain Rule with $f(x) = x^{10}$ and $g(x) = 5x - 3$.

$$f(x) = x^{10} \qquad f'(x) = 10x^9$$
$$g(x) = 5x - 3 \qquad g'(x) = 5$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 10(g(x))^9 \cdot 5 \\ &= 50(5x-3)^9 \end{aligned}$$

5. Use the Chain Rule with $f(x) = x^{-10}$ and $g(x) = 5x - 3$.

$$f(x) = x^{-10} \qquad f'(x) = -10x^{-11}$$
$$g(x) = 5x - 3 \qquad g'(x) = 5$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= -10(g(x))^{-11} \cdot 5 \\ &= -50(5x-3)^{-11} \\ &= -\frac{50}{(5x-3)^{11}} \end{aligned}$$

Note that some resources may leave the exponent as negative, while others will rewrite with a positive exponent in the denominator.

6. First, combine exponents.

$$\begin{aligned} x^3\sqrt{x} &= x^3x^{1/2} \\ &= x^{7/2} \end{aligned}$$

Then use the Power Rule.

$$\begin{aligned} h'(x) &= \frac{7}{2}x^{7/2-1} \\ &= \frac{7}{2}x^{5/2} \end{aligned}$$

7. Use the Product Rule with $f(x) = \sin(x)$ and $g(x) = \sqrt{x}$.

$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$
$$g(x) = \sqrt{x} \qquad g'(x) = \frac{1}{2\sqrt{x}}$$

Then

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= \sin(x) \cdot \frac{1}{2\sqrt{x}} + \sqrt{x} \cdot \cos(x) \\ &= \frac{\sin(x)}{2\sqrt{x}} + \sqrt{x}\cos(x) \end{aligned}$$

8. Use the Quotient Rule.

$$f(x) = ax + b \qquad f'(x) = a$$
$$g(x) = ax - b \qquad g'(x) = a$$

$$\begin{aligned}
\frac{d}{dx}\frac{ax+b}{ax-b} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\
&= \frac{(ax-b)\cdot a - (ax+b)\cdot a}{(ax-b)^2} \\
&= \frac{a^2x - ab - (a^2x + ab)}{(ax-b)^2} \\
&= \frac{a^2x - ab - a^2x - ab}{(ax-b)^2} \\
&= -\frac{2ab}{(ax-b)^2}
\end{aligned}$$

9. Use the Product Rule with $f(x) = \sin(x)$ and $g(x) = \cos x$.

$$f(x) = \sin(x) \qquad f'(x) = \cos(x)$$
$$g(x) = \cos(x) \qquad g'(x) = -\sin(x)$$

Then

$$\begin{aligned}
h'(x) &= f(x)g'(x) + g(x)f'(x) \\
&= \sin(x)(-\sin(x)) + \cos(x)\cdot\cos(x) \\
&= \cos^2(x) - \sin^2(x)
\end{aligned}$$

10. Use the Chain Rule.

$$f(x) = \sqrt{x} \qquad f'(x) = \frac{1}{2\sqrt{x}}$$
$$g(x) = 3x - 5 \qquad g'(x) = 3$$

Then

$$\begin{aligned}
h'(x) &= f'(g(x))g'(x) \\
&= \frac{1}{2\sqrt{g(x)}}\cdot 3 \\
&= \frac{3}{2\sqrt{3x-5}}.
\end{aligned}$$

11. Use the Chain Rule.

$$f(x) = \cos(x) \qquad f'(x) = -\sin(x)$$
$$g(x) = x^3 \qquad g'(x) = 3x^2$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= -\sin(g(x)) \cdot 3x^2 \\ &= -3x^2 \sin(x^3). \end{aligned}$$

12. Use the Chain Rule. Remember the notation: $\cos^3(x) = (\cos(x))^3$.

$$\begin{aligned} f(x) &= x^3 & f'(x) &= 3x^2 \\ g(x) &= \cos(x) & g'(x) &= -\sin(x) \end{aligned}$$

Then

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) \\ &= 3(g(x))^2(-\sin(x)) \\ &= -3\sin(x)\cos^2(x). \end{aligned}$$

13. Write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and use the Quotient Rule.

$$\begin{aligned} f(x) &= \sin(x) & f'(x) &= \cos(x) \\ g(x) &= \cos(x) & g'(x) &= -\sin(x) \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\frac{\sin(x)}{\cos(x)} &= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \\ &= \frac{\cos(x) \cdot \cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x). \end{aligned}$$

Here, we used the identity $\cos^2(x) + \sin^2(x) = 1$ and the definition $\sec(x) = \dfrac{1}{\cos(x)}$.

14. Rewrite $\dfrac{\sin(x)}{x^{-3}} = x^3 \sin(x)$ and use the Product Rule.

$$\begin{aligned} f(x) &= x^3 & f'(x) &= 3x^2 \\ g(x) &= \sin(x) & g'(x) &= \cos(x) \end{aligned}$$

Then

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^3 \cdot \cos(x) + \sin(x) \cdot 3x^2 \\ &= x^3 \cos(x) + 3x^2 \sin(x) \\ &= x^2(x \cos(x) + 3 \sin(x)). \end{aligned}$$

You do not have to factor out in the last step, but be aware that when using other resources, answers may be written this way.

15. (a) Use the Chain Rule.
 (b) Rewrite as $3x^{-7}$ and use the Power Rule.
 (c) Rewrite as $x^2 \cos(x)$ and use the Product Rule.
 (d) Combine exponents to get $x^{5/2}$ and use the Product Rule.
 (e) Simplify by dividing and use the Power Rule.
 (f) Use the Quotient Rule.
 (g) Use the Chain Rule.
 (h) Rewrite as $3x^4$ and use the Power Rule.

Chapter 4

The Second Derivative

4.1 What happens when $f'(x) = 0$?

In Chapter 2, we began our study of the algebra and geometry of derivatives. We saw that:

> If $f'(x) > 0$ for some value of x, then the function $f(x)$ is increasing at that value of x.
>
> If $f'(x) < 0$ for some value of x, then the function $f(x)$ is decreasing at that value of x.

But what happens when $f'(x) = 0$? This is a more complicated scenario, illustrated in Figure 4.1.

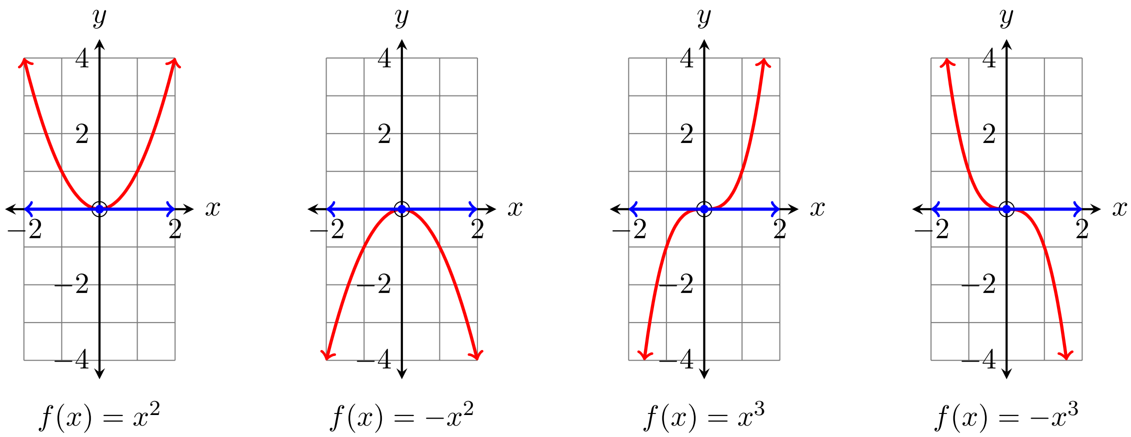


Figure 4.1: What can happen when $f'(x) = 0$.

Let's look at these graphs in detail. In each graph, the blue line is tangent at $x = 0$, and is a horizontal line with slope 0, and so $f'(0) = 0$ in each case.

1. For $f(x) = x^2$, we have what is called a **local minimum** at $x = 0$. This means as we go from left to right, the function decreases until it hits $(0, 0)$, and then starts increasing.
2. For $f(x) = -x^2$, we have what is called a **local maximum** at $x = 0$. This means as we go from left to right, the function increases until it hits $(0, 0)$, and then starts decreasing.
3. For $f(x) = x^3$, we have what is called an **inflection point,** or a **point of inflection.** In this case the function keeps increasing as we pass through $(0, 0)$.
4. For $f(x) = -x^3$, we also have an inflection point, but the function keeps decreasing as we pass through $(0, 0)$.

How do we know which is which? Yes, we can look at the graph. But to decide *without* a graph, we have to use calculus. In this section, we'll learn to use **sign charts** for $f'(x)$ to make this decision, and in the next section, we'll look at how to use **second derivatives.**

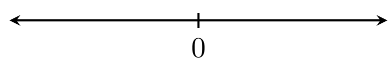
Making Sign Charts

We'll first look at the far left graph in Figure 4.1, $f(x) = x^2$. To make a sign chart for $f'(x)$:

1. Find all values of x where $f'(x) = 0$;
2. Plot these values on a number line;
3. This divides the line into intervals – choose *one* point from each interval (one that is easy to evaluate) and evaluate $f'(x)$; if $f'(x) > 0$, write "+" over the interval, and if $f'(x) < 0$, write "−" above the interval. We'll first go through these steps, and then interpret the results.

Since $f(x) = x^2$, then using the Power Rule, we get $f'(x) = 2x$.

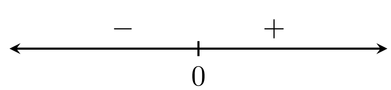
1. If $f'(x) = 2x = 0$, then $x = 0$.
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f'(-1) &= 2(-1) \\ &= -2 \\ &< 0 \\ f'(1) &= 2(1) \\ &= 2 \\ &> 0. \end{aligned}$$

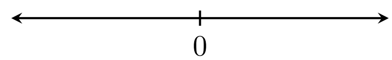
This yields the following number line:



How do we interpret this? On the interval $(-\infty, 0)$, $f'(x)$ is negative, and so we know the function is decreasing on this interval. But on $(0, \infty)$, we see that $f'(x)$ is positive, and so the function is increasing on this interval. Because we go from decreasing to increasing as we pass through $x = 0$, this means there must be a local minimum at $x = 0$. This can be confirmed by looking at the graph.

Now let's look at the rightmost graph in Figure 4.1, $f(x) = -x^3$. We'll make a sign chart here as well. Since $f(x) = -x^3$, then using the Power Rule, we get $f'(x) = -3x^2$.

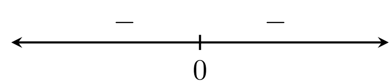
1. If $f'(x) = -3x^2 = 0$, then $x = 0$.
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f'(-1) &= -3(-1)^2 \\ &= -3 \\ &< 0 \\ f'(1) &= -3(1)^2 \\ &= -3 \\ &< 0. \end{aligned}$$

   This yields the following number line:



How do we interpret this? On the interval $(-\infty, 0)$, $f'(x)$ is negative, and so we know the function is decreasing on this interval. But on $(0, \infty)$, we see that $f'(x)$ is also negative, and so the function is decreasing on this interval as well. Because we continuously decrease as we pass through $x = 0$, this means there must be an inflection point at $x = 0$. This can also be confirmed by looking at the graph.

The algebra for these examples was fairly easy, but the point is to introduce the concepts. This is one more example of a recurring theme: we make informal observations about a function by looking at its graph, and then we back up our observations using calculus.

Example 1

Now we'll look at a little more complicated example. Consider the function $f(x) = x^3 - 3x + 2$, shown in Figure 4.2.

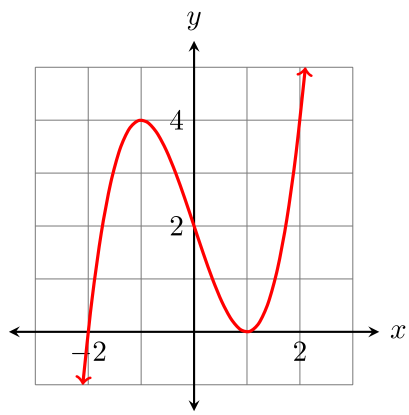


Figure 4.2: Graph of $f(x) = x^3 - 3x + 2$.

It looks like there is a local maximum at $x = -1$ and a local minimum at $x = 1$. Let's verify this by making a sign chart.

1. Since $f(x) = x^3 - 3x + 2$, then using the Power Rule,

$$f'(x) = 3x^2 - 3.$$

Then solving $f'(x) = 0$:

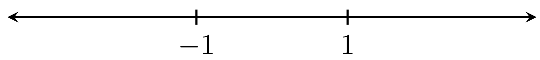
$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 3 &= 0 \\ 3(x^2 - 1) &= 0 \\ 3(x+1)(x-1) &= 0 \\ x &= 1 \\ x &= -1 \end{aligned}$$

This isn't the only way to solve. You can do the following.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 3 &= 0 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

WARNING!!! Be very careful if you do this. A common mistake is to forget $x = -1$ when doing it this way. That is, you just go from $x^2 = 1$ to $x = 1$ and leave out the other value of x. I've seen this happen **many** times.

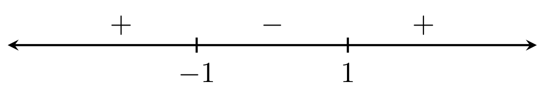
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -2$, $x = 0$, and $x = 2$.

$$\begin{aligned} f'(-2) &= 3((-2)^2) - 3 \\ &= 9 \\ &> 0 \\ f'(0) &= 3(0^2) - 3 \\ &= -3 \\ &< 0 \\ f'(2) &= 3(2^2) - 3 \\ &= 9 \\ &> 0. \end{aligned}$$

This yields the following number line:



So at $x = -1$, the graph changes from increasing to decreasing, and so there is a local maximum there. And at $x = 1$, the graph changes from decreasing to increasing, and so there is a local minimum there.

Assessment Expectation: When asked to find local maxima and minima (plurals of maximum and minimum), ALWAYS include the y-values and write your answer as a point. Since $f(-1) = 4$ and $f(1) = 0$, you would say there is local maximum at $(-1, 4)$ and a local minimum at $(1, 0)$.

Summary:

If $f'(x) > 0$ for some value of x, then the function $f(x)$ is increasing at that value of x.

If $f'(x) < 0$ for some value of x, then the function $f(x)$ is decreasing at that value of x.

If $f'(x) = 0$, there may be a local minimum, a local maximum, or an inflection point (determined by making a sign chart).

Homework

1. Consider the graph of $f(x) = x^3 - 12x - 3$. By making a sign chart for $f'(x)$, find all local minima and maxima. Visually verify this by graphing on desmos.

2. Consider the graph of $f(x) = \sin(x)$ on the interval $[0, 2\pi]$. By making a sign chart for $f'(x)$, find all local minima and maxima. Visually verify this by graphing on desmos.

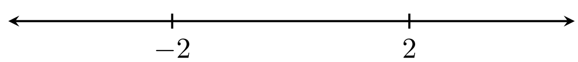
Solutions

1. Since $f(x) = x^3 - 12x - 3$, then $f'(x) = 3x^2 - 12$.

   (a) To make a sign chart, we need to know where $f'(x) = 3x^2 - 12 = 0$.

$$
\begin{aligned}
f'(x) &= 0 \\
3x^2 - 12 &= 0 \\
3(x^2 - 4) &= 0 \\
3(x+2)(x-2) &= 0 \\
x &= 2 \\
x &= -2
\end{aligned}
$$

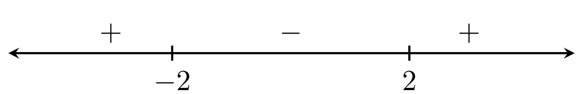
   (b) This gives the following number line:



   (c) Now choose one value from each interval. Easy values are $x = -3$, $x = 0$, and $x = 3$.

$$
\begin{aligned}
f'(-3) &= 3((-3)^2) - 12 \\
&= 15 \\
&> 0 \\
f'(0) &= 3(0^2) - 12 \\
&= -12 \\
&< 0 \\
f'(2) &= 3(3^2) - 12 \\
&= 15 \\
&> 0.
\end{aligned}
$$

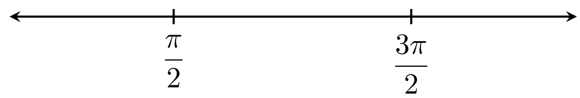
   This yields the following number line:



Because $f'(x)$ goes from $+$ to $-$ at -2, $f(x)$ increases and then decreases. Since $f(-2) = 13$, there is a local maximum at $(-2, 13)$. Because $f'(x)$ goes from $-$ to $+$ at 2, $f(x)$ decreases and then increases. Since $f(2) = -19$, there is a local minimum at $(2, -19)$.

2. To make a sign chart, we need to know where $\cos(x) = 0$ on the interval $[0, 2\pi]$. We can look at the unit circle and see where the x-value is 0. This occurs when $x = \pi/2$ and $x = 3\pi/2$.

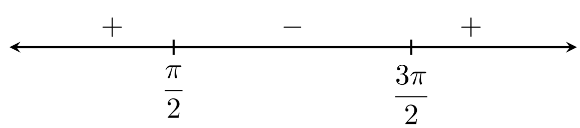
   (a) This gives the following number line:

   (b) Looking at easy values in each interval, we choose $x = \frac{\pi}{4}, \pi, \frac{7\pi}{4}$. Then

$$\begin{aligned}
f'(\pi/4) &= \cos(\pi/4) \\
&= 1/\sqrt{2} \\
&> 0 \\
f'(\pi) &= \cos(\pi) \\
&= -1 \\
&< 0 \\
f'(7\pi/4) &= \cos(7\pi/4) \\
&= 1/\sqrt{2} \\
&> 0
\end{aligned}$$

   This yields the following number line:

   Since $f'(x)$ goes from + to − at $\pi/2$, $f(x)$ increases and then decreases. So there is a local maximum at $(\pi/2, 1)$. Since $f'(x)$ goes from − to + at $3\pi/2$, $f(x)$ decreases and then increases. So there is a local minimum at $(3\pi/2, -1)$.