Final Exam, MA 131M, 12 December 2022

Please show as much work as you need. Keep in mind that if you skip several steps and get an incorrect answer, it will be very difficult to assign partial credit. Please take a moment to skim through the test first and start with the questions you feel most confident about.

1. (10) Using the definition of the derivative, find $f'(x)$ if $f(x) = 3x^2 - 5x$.

$$f(x+h) = 3(x+h)^2 - 5(x+h) = 3(x^2+2xh+h^2) - 5x - 5h$$
$$= 3x^2 + 6xh + 3h^2 - 5x - 5h$$

$$\frac{f(x+h)-f(x)}{h} = \frac{3x^2+6xh+3h^2-5x-5h-(3x^2-5x)}{h}$$
$$= \frac{\cancel{3x^2}+6xh+3h^2\cancel{-5x}-5h\cancel{-3x^2}\cancel{+5x}}{h}$$
$$= \frac{6xh+3h^2-5h}{h} = \frac{h(6x+3h-5)}{h}$$
$$= 6x+3h-5$$

$$\lim_{h\to 0}(6x+3h-5) = 6x+3\cdot 0-5 = 6x-5$$

2. Take the derivatives of the following functions. Simplify as much as possible.

(a) (4) $h(x) = \sqrt{x^3} - \frac{5}{x^6}$ $= x^{\frac{3}{2}} - 5x^{-6}$

$$h'(x) = \frac{3}{2}x^{\frac{1}{2}} - 5(-6x^{-7})$$
$$= \frac{3}{2}\sqrt{x} + \frac{30}{x^7}$$

(b) (4) $h(x) = x^3 \ln x$

$f(x) = x^3 \qquad f'(x) = 3x^2$

$g(x) = \ln x \qquad g'(x) = \frac{1}{x}$

$f(x)\,g'(x) + g(x)\,f'(x)$

$x^3 \cdot \frac{1}{x} + \ln x\,(3x^2) = x^2 + 3x^2 \ln x$

(c) (4) $h(x) = \dfrac{4 - bx}{4 + bx}$

$f(x) = 4 - bx \qquad f'(x) = -b$

$g(x) = 4 + bx \qquad g'(x) = b$

$$\frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \frac{(4+bx)(-b) - (4-bx)b}{(4+bx)^2}$$

$$= \frac{-4b - \cancel{b^2x} - 4b + \cancel{b^2x}}{(4+bx)^2} = \frac{-8b}{(4+bx)^2}$$

(d) (4) $h(x) = \tan(x^3)$

$f(x) = \tan(x) \qquad f'(x) = \sec^2(x)$

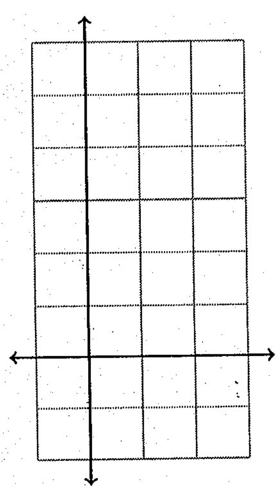
$g(x) = x^3 \qquad g'(x) = 3x^2$

$f'(g(x))g'(x)$

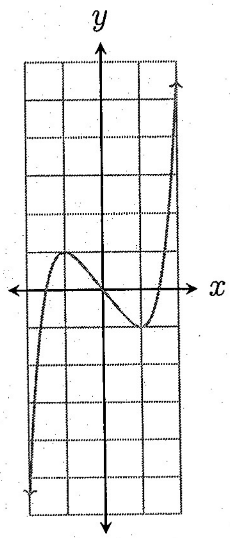
$\sec^2(g(x)) \cdot 3x^2 = 3x^2 \sec^2(x^3)$

3. (10) Consider the following piecewise-defined function. Assume c is a constant. What must be the value of c so that $g(x)$ is a continuous function? Sketch a graph of this function on the interval $[-1, 3]$ below.

$$g(x) = \begin{cases} cx + 2, & x \leq 1, \\ x - 2, & x > 1. \end{cases}$$



4. (12) Find the global extrema of the function $f(x) = \frac{1}{4}(x^5 - 5x)$ on the closed interval $[-2, 2]$. The graph is for reference *only.* You *must* perform all of the calculus calculations for full credit.



$f'(x) = \frac{1}{4}(5x^4 - 5) = 0$

$5x^4 - 5 = 0$

$5x^4 = 5$

$x^4 = 1$

$x = \pm 1$

$f(-2) = \frac{1}{4}((-2)^5 - 5(-2)) = -\frac{11}{2}$

$f(-1) = \frac{1}{4}((-1)^5 - 5(-1)) = 1$

$f(1) = \frac{1}{4}(1^5 - 5(1)) = -1$

$f(2) = \frac{1}{4}(2^5 - 5 \cdot 2) = \frac{11}{2}$

Global max: $(2, \frac{11}{2})$

Global min: $(-2, -\frac{11}{2})$

5. (12) Suppose you throw a baseball at an angle of 60° from the horizontal at a speed of 25 m/s. When the baseball leaves your hand, it is 1.8 m above the ground.

(a) Write the displacement equations. No need to solve.

(b) The ball will hit the ground after 4.5 seconds. How far away will the ball land?

6. (5) Find $\lim_{x\to 0^-} \dfrac{5x-1}{x^2\tan(x)}$.   Of the form $\frac{-1}{0}$. So DNE

$+ \quad -$

DNE $(+\infty)$

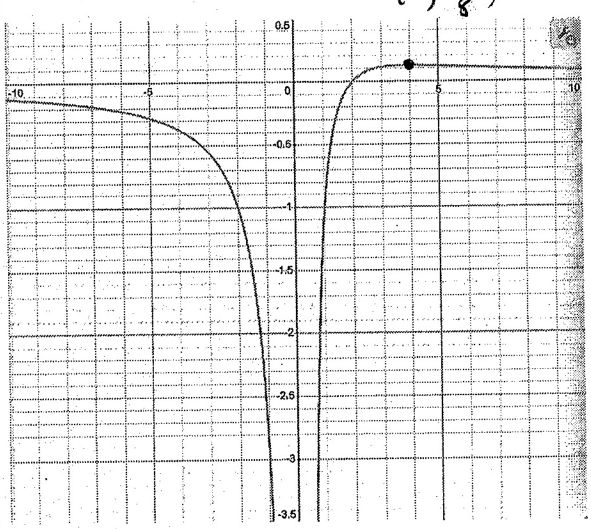
7. (12) Consider $f(x) = \dfrac{x-2}{x^2}$. Here are the derivatives:

$$f'(x) = \frac{4-x}{x^3}, \quad f''(x) = \frac{2x-12}{x^4}.$$

(a) Find the x-values and y-values of the local maxima and minima. Label them on the graph. You do **not** have to show they are maxima or minima. Just label them on the graph.

(b) By creating a **sign chart**, find the intervals where $f(x)$ is concave up and concave down.

$(4, \frac{1}{8})$



VA at $x=0$

a) $f'(x) = 0$

$\dfrac{4-x}{x^3} = 0$

$4 - x = 0$

$x = 4$

$f(4) = \dfrac{4-2}{4^2} = \dfrac{1}{8}$

b) $f''(x) = 0$

$\dfrac{2x-12}{x^4} = 0$

$2x - 12 = 0$

$2x = 12$

$x = 6$

$- \quad\quad - \quad\quad +$

$0 \quad\quad 6$

$f''(-1) = \dfrac{2(-1)-12}{(-1)^4} = -14 < 0$

$f''(1) = \dfrac{2(1)-12}{1^4} = -10 < 0$

$f''(7) = \dfrac{2\cdot 7-12}{7^4} > 0$

CU: $(6, \infty)$

CD: $(-\infty, 0) \cup (0, 6)$

8. (5) Find $\int \left(\frac{2}{x^3} - \frac{3}{x}\right) dx$.

$$\int \left(2x^{-3} - \frac{3}{x}\right) dx = 2 \cdot \frac{1}{-2} x^{-2} - 3\ln x + c$$

$$= -\frac{1}{x^2} - 3\ln x + C$$

9. (10) Solve the initial value problem $f'(x) = 5\sin(x) - 6\cos(x)$, $f(\pi) = 7$.

$$f(x) = \int f'(x)\,dx = \int (5\sin(x) - 6\cos(x))\,dx$$

$$= -5\cos(x) - 6\sin(x) + C$$

$$f(\pi) = -5\cos(\pi) - 6\sin(\pi) + C = 7$$

$$-5(-1) - 6 \cdot 0 + C = 7$$

$$C = 2$$

$$f(x) = 5\sin(x) - 6\cos(x) + 2$$

10. (10) Find an area $A(x)$ function using $x_0 = -1$ and $f(x) = x^3 - 2x$. Verify that $A'(x) = f(x)$.

$$A(x) = \int_{-1}^{x} (u^3 - 2u)\,du = \frac{1}{4}u^4 - u^2 \Big|_{-1}^{x}$$

$$= \frac{1}{4}x^4 - x^2 - \left(\frac{1}{4}\cdot(-1)^4 - (-1)^2\right)$$

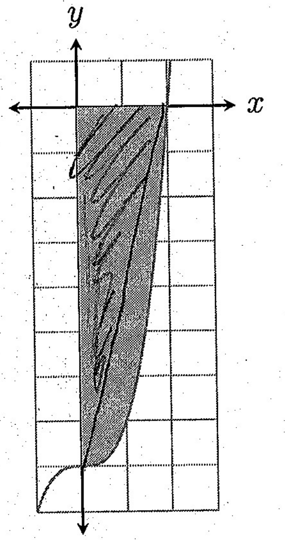
$$= \frac{1}{4}x^4 - x^2 - \frac{1}{4} + 1 = \frac{1}{4}x^4 - x^2 + \frac{3}{4}$$

$$A'(x) = \frac{1}{4} \cdot 4x^3 - 2x = x^3 - 2x$$

11. (15) Consider the area between the function $f(x) = x^3 - 8$ and the x-axis on the interval $[0, 2]$.

   (a) Draw a triangle in the shaded region which best approximates the area, and create a guesstimate from the area of this triangle.

   (b) Calculate the exact area of this region.



a) $\frac{1}{2} \cdot 8 \cdot 2 = 8$, below x-axis: -8

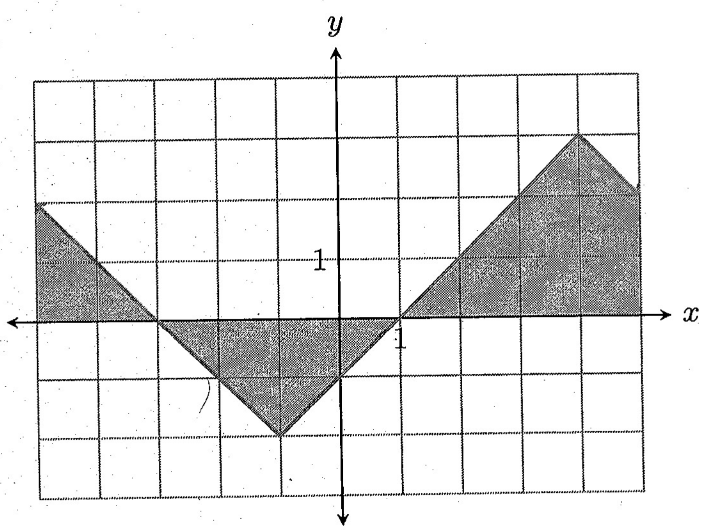
b) $\int_0^2 (x^3 - 8)\,dx = \frac{1}{4}x^4 - 8x \Big|_0^2$

$= \frac{1}{4} \cdot 2^4 - 8 \cdot 2 - \left(\frac{1}{4} \cdot 0^4 - 8 \cdot 0\right)$

$= 4 - 16$

$= -12$

12. (16) Consider the function $f(x)$, graphed below. Find the following.



(a) $\int_{-5}^{5} f(x)\,dx = 5$

(b) $\int_{1}^{-5} f(x)\,dx = 2$

(c) $\int_{3}^{-2} f(x)\,dx = \frac{3}{2}$

(d) $\int_{-4}^{2} f(x)\,dx = -3$

13. (8) Find $\int 6x\cos(x^2)\,dx$.

$u = x^2$

$du = 2x\,dx \Rightarrow dx = \frac{1}{2x}\,du$

$$\int 6x \cdot \cos(x^2)\,dx = \int 6x \cdot \cos(u) \cdot \frac{du}{2x}$$

$$= \int 3\cos(u)\,du$$

$$= 3\sin(u) + C$$

$$= 3\sin(x^2) + C$$

14. (8) Find $\int \frac{2}{\sqrt{1-16x^2}}\,dx$.

$u = 4x$

$du = 4\,dx \rightarrow dx = \frac{1}{4}\,du$

$$\int \frac{2}{\sqrt{1-16x^2}}\,dx = \int \frac{2}{\sqrt{1-(4x)^2}}\,dx$$

$$= \int \frac{2}{\sqrt{1-u^2}} \cdot \frac{1}{4}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{1}{2}\arcsin(u) + C$$

$$= \frac{1}{2}\arcsin(4x) + C$$

15. (5) Find $\frac{d}{dx}\int_x^4 e^{\overset{u}{\cancel{x}}} \cos(\overset{u}{\cancel{x}})\, dx.$ $= -\frac{d}{dx}\int_4^x e^u \cos(u)\, du$

$= -e^x \cos(x)$

Extra Credit: Explain the Fundamental Theorem of Calculus, Parts I and II.