1 What Calclulus is All About

We will look at the basic concepts behind calculus by studying three examples. Much of calculus comes from physics, so we'll focus our attention on velocity and displacement. In physics, velocity and displacement can be positive or negative, so we use these terms instead of speed and distance.

Example 1

The graphs below represent a road trip you might go on. You drive at a constant rate of 20 km/hr for 4 hours, and so $v(t) = 20$.
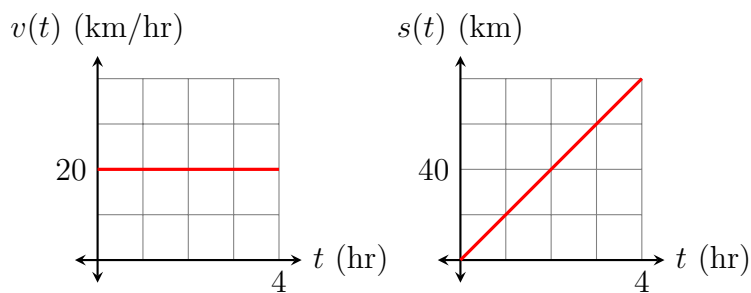
Figure 1: Velocity graph (left), and displacement graph (right).

Since displacement $=$ rate $\times$ time, then $s(t) = 20t$. This is graphically represented below by the blue rectangle.
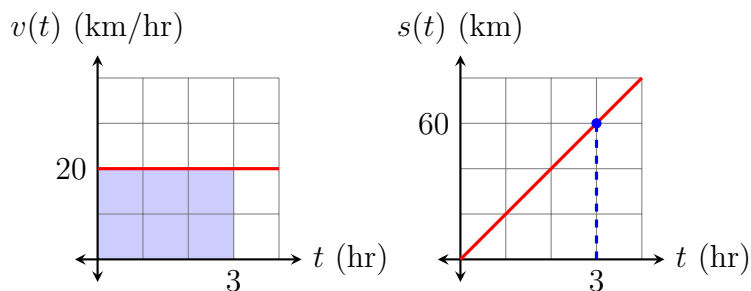
Figure 2: Velocity graph (left), and displacement graph (right).

Let's look at what happens after driving for three hours. You've traveled a total of $s(3) = 60$ km, which is represented by the blue rectangle; that is, the area under $v(t)$ up to three hours. Also, the slope of the line at $(3, 60)$ on the displacement curve is $v(3)$, which is 20 km/hr. It might look like the slope is 1, but remember that the units on the axes are different.

15 Example 2

16 Now we'll take a look at a road trip where your velocity is not constant. Since you are
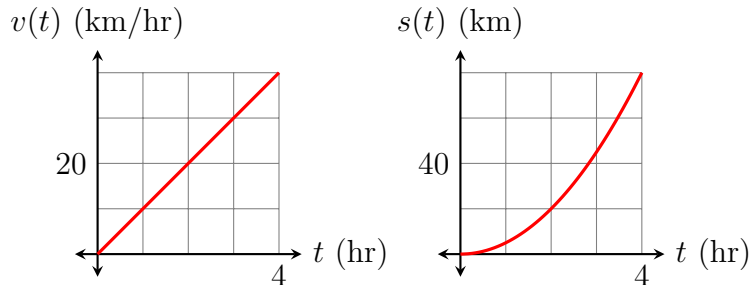17 driving at 40 km/hr after 4 hours, your velocity is given by $v(t) = 10t$.

Figure 3: Velocity graph (left), and displacement graph (right).

18 Since your velocity is not constant, we can't use displacement = rate × time. But we can
19 still use the fact that $s(t)$ is the area under $v(t)$ up to t hours, just as in Example 1.
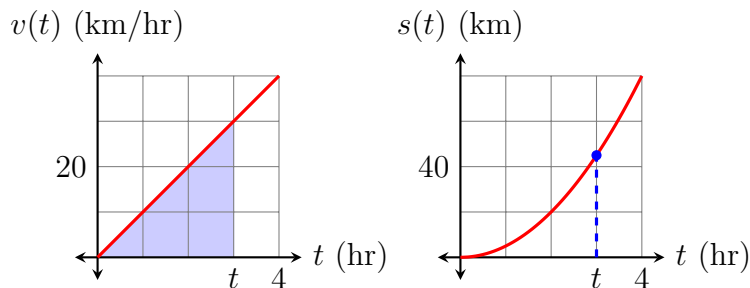

Figure 4: Velocity graph (left), and displacement graph (right).

Since the blue area is a triangle, we can use $A = \dfrac{1}{2}bh$, so

$$s(t) = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot t \cdot 10t = 5t^2.$$

20 This function is graphed on the right; it is a parabola.

21 In Example 1, we saw that the slope of the line at $(3, s(3))$ was 20 km/hr. In this example,
22 $s(3) = 45$. But what is the slope of the parabola at $(3, 45)$? Now we've entered calculus
23 territory. We're not only interested in the slope of a *line,* we're interested in the slope of a
24 *curve.*

25 We can accomplish this by looking at the tangent line to a curve. Below (middle graph), you
26 can see that the blue line intersects the parabola *only* at the point $(3, 45)$. In geometry, when
27 a line intersects a curve at just one point, we call this a *tangent line.* When you zoom in on
28 the blue box (right graph), you see that near the point $(3, 45)$, it's hard to tell the difference
29 between the tangent line and the parabola. This is an important property in calculus.

30 What is the slope of this tangent line? Since $v(3) = 30$, this tangent line has a slope of 30
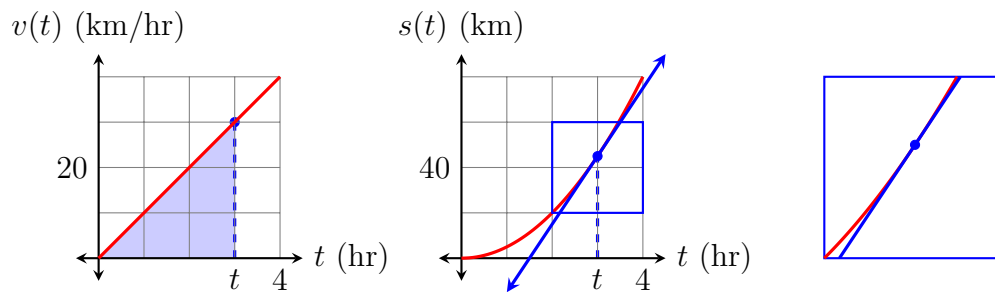km/hr.

Figure 5: Velocity graph (left), tangent line (middle), zooming in (right).

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32 We've just seen the most fundamental concept in calculus.

33

> The area underneath the velocity curve up to time t corresponds to the displacement at time t, and the slope of the displacement curve at time t is the velocity at time t.
>
> Using function notation, we would say that the area up to t underneath $v(t)$ is $s(t)$, and the slope of the tangent line at $(t, s(t))$ is given by $v(t)$.

₃₄ **Example 3**

₃₅ Let's look at another road trip. This time, you start out at 20 km/hr, but you slow down at
₃₆ a constant rate. After 2 hours, you turn around and start driving in the opposite direction.
₃₇ This is why we use *velocity* instead of *speed.* If you are driving east with a positive velocity,
₃₈ it means that if you turn around and drive west, your velocity is negative.
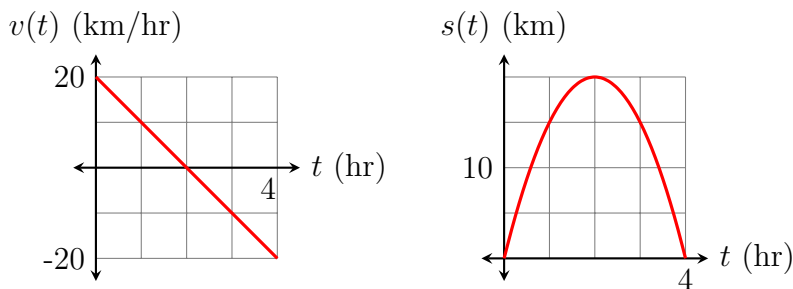

Figure 6: Velocity graph (left), displacement graph (right).

₃₉ Since you start out at 20 km/hr and end up at -20 km/hr after 4 hours (meaning you end
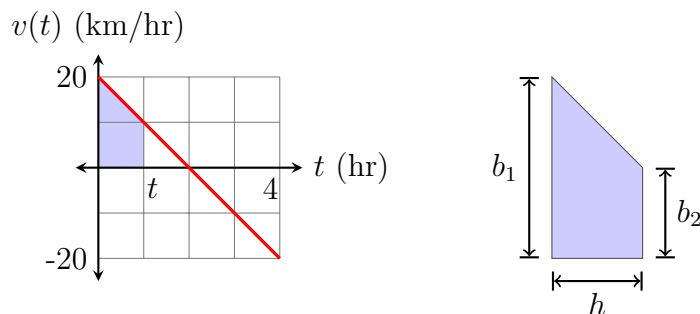₄₀ up driving in the opposite direction), then $v(t) = 20 - 10t$.

Figure 7: Area under the velocity graph (left), area of the trapezoid (enlarged, right).

As in Example 2, we can find the displacement curve by looking at an area. This time, the
area is a trapezoid, so we need the formula $A = \dfrac{1}{2}(b_1 + b_2)h$ from geometry. In our example,
b_1 is always 20, b_2 corresponds to $v(t)$, and h is just t. This means that

$$s(t) = \frac{1}{2} \cdot (b_1 + b_2) \cdot h = \frac{1}{2} \cdot (20 + v(t)) \cdot t = \frac{1}{2}(20 + (20 - 10t)) \cdot t = 20t - 5t^2.$$

41 Now look at the graph below. Since $v(t)$ is positive here, the slope of the tangent line to the curve $s(t)$ is positive.
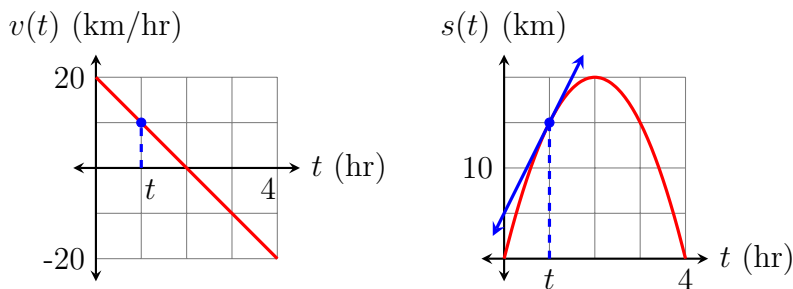

Figure 8: Velocity graph (left), displacement graph (right).

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43 But in the next graph, the value of $v(t)$ is negative, and you can see that the slope of the
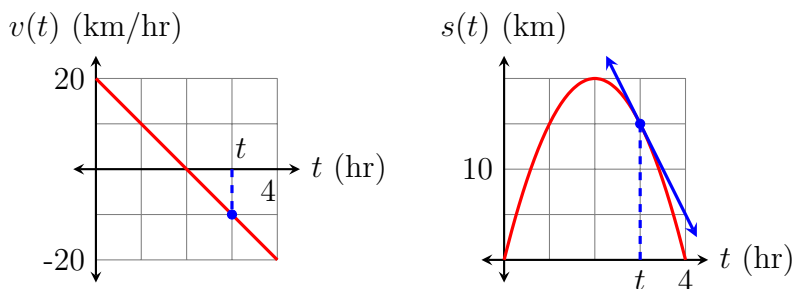44 tangent line to the curve is negative.

Figure 9: Velocity graph (left), displacement graph (right).

45 Now when $t = 4$, you can see that $s(4) = 0$. This represents the area under the velocity
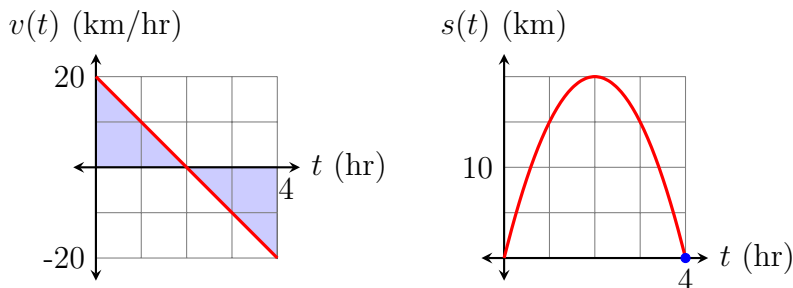46 curve up to $t = 4$. How can this area be zero?

Figure 10: Velocity graph (left), displacement graph (right).

47 In calculus, areas can be *negative.* Let's say when your velocity is positive, you're driving
48 east, so when your velocity is negative, you're driving west. Now the area *below* the velocity
49 curve when the velocity is *positive* is how far you traveled east. But the area *above* the velocity

50 curve when the velocity is *negative* is how far you traveled west – and this area is *negative.*
51 Looking at the graph on the left, the triangles are congruent, but one has positive area and
52 the other has negative area. Their areas cancel out, which is why your total displacement is
53 0 – you've traveled just as far while driving east as you did while driving west.

54 So considering velocity and displacement, instead of speed of distance, is very important in
55 calculus.

56

> When your velocity is positive, the corresponding slope of the tangent line on the displacement curve will be positive.
>
> When your velocity is negative, you're traveling in the opposite direction, and the corresponding slope of the tangent line on the displacement curve will be negative.
>
> The area corresponding to traveling with a positive velocity will always be positive, so your displacement will be positive.
>
> The area corresponding to traveling with a negative velocity in the opposite direction will always be negative, so your displacement will be negative.