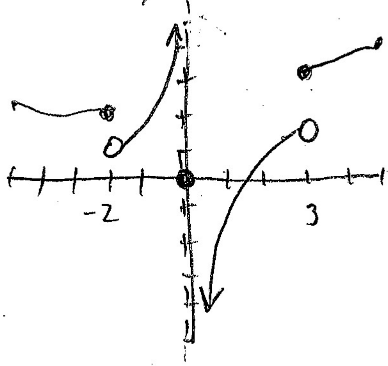
SOLUTIONS, EXAM I

DAY 16
9 MAR 22

1) $\lim_{x\to 2} \dfrac{\frac{x}{2}-\frac{2}{x}}{x-2} = \lim_{x\to 2} \dfrac{\frac{x}{2}\cdot\frac{x}{x}-\frac{2}{x}\cdot\frac{2}{2}}{x-2} = \lim_{x\to 2} \dfrac{\frac{x^2-4}{2x}}{x-2}$

$= \lim_{x\to 2} \dfrac{x^2-4}{2x(x-2)} = \lim_{x\to 2} \dfrac{(x+2)\cancel{(x-2)}}{2x\cancel{(x-2)}} = \lim_{x\to 2} \dfrac{x+2}{2x} = \underline{1}$

2) Many answers are possible.



3) $f(x)$ is continuous everywhere, except possibly $x=1$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} |x| = 1$, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 = 1$,

$f(1) = |1| = 1$. Since all these are equal, f is continuous at $x=1$, and so f is continuous.

4) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{x\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0} \dfrac{(\cancel{x}+h) + \cancel{\sqrt{x+h}\sqrt{x}} - \cancel{\sqrt{x}\sqrt{x+h}} - \cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$

$= \lim_{h\to 0} \dfrac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \lim_{x\to 0} \dfrac{1}{\sqrt{x+h}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

Domain of $f'(x) = (-\infty, 0) \cup (0, \infty)$.

5) a) $f(x) = 3x^{-2} - 5x^{3} + x^{\frac{3}{2}}$

$f'(x) = 3(-2x^{-3}) - 5(3x^{2}) + \frac{3}{2}x^{\frac{1}{2}}$

$= -6x^{-3} - 15x^{2} + \frac{3}{2}x^{\frac{1}{2}}$

b) $g'(x) = \sec^2(x) - 4\cos(x)$

c) $p'(x) = x^2 \frac{d}{dx}(\cos(x)) + \cos(x)\frac{d}{dx}(x^2)$

$= -x^2 \sin x + 2x\cos(x)$

d) $q'(x) = \dfrac{(3-2x)\frac{d}{dx}(2-x) - (2-x)\frac{d}{dx}(3-2x)}{(3-2x)^2}$

$= \dfrac{(3-2x)(-1) - (2-x)(-2)}{(3-2x)^2} = \dfrac{-3+2x+4-2x}{(3-2x)^2}$

$= \dfrac{1}{(3-2x)^2}$

6) $\lim_{x\to 2} \dfrac{1-x^3}{(x-2)^2}$ $\begin{matrix} \leftarrow 1-2^3 = -7 \\ \leftarrow 0 \end{matrix}$

Thus, the limit does not exist

Near $x=2$: $\dfrac{1-x^3}{(x-2)^2}$ $\begin{matrix} \leftarrow \text{neg} \\ \leftarrow \text{pos} \end{matrix}$ $\quad \dfrac{\text{neg}}{\text{pos}} = \text{neg}$, so $\lim_{x\to 2}\dfrac{1-x^3}{(x-2)^2}$ DNE $(-\infty)$

The graph has a vertical asymptote at $x = 2$.

7) $f(x) = |x|$ is <u>not</u> differentiable at 0. But using Cal's definition,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0-h)}{2h} = \lim_{h\to 0} \frac{|h| - |-h|}{2h}$$

$$= \lim_{h\to 0} \frac{|h| - |h|}{2h} = \lim_{h\to 0} \frac{0}{2h} = 0.$$

So Cal's idea is not a good one, since his definition would allow $f(x) = |x|$ to be differentiable at 0.

8) Using the product rule:

$$\frac{d}{dx}[f(x)]^2 = \frac{d}{dx} f(x)f(x) = f(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}f(x)$$

$$= f(x)f'(x) + f(x)f'(x) = 2f(x)f'(x)$$

9) Yes, it is possible.

$\lim_{x\to 0} \frac{1}{x}$ does <u>not</u> exist. $\lim_{x\to 0} x = 0$, so this limit exists. But $\lim_{x\to 0} \frac{1}{x} \cdot x = \lim_{x\to 0} 1 = 1$. So one of the limits does <u>not</u> exist, but the limit of the product <u>does</u> exist.