1. Find the numbers at which the function

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 & \text{if } 1 \le x < 3 \\ x+5 & \text{if } x \ge 3 \end{cases}$$

is discontinuous. At which of these points is f continuous from the right, from the left, or neither?

Check $x = 1, 3$.

$x=1$: $\lim_{x\to1^-} f(x) = \lim_{x\to1^-} x = 1$, $\lim_{x\to1^+} f(x) = \lim_{x\to1^+} x^2 = 1$.

$f(1) = 1 \Rightarrow$ continuous at $x = 1$.

$x=3$: $\lim_{x\to3^-} f(x) = \lim_{x\to3^-} x^2 = 9$, $\lim_{x\to3^+} f(x) = \lim_{x\to3^+} (x+5) = 8$

$f(3) = 8$. Discontinuous at $x = 3$, but continuous from the right.

2. Use the Intermediate Value Theorem to show that there is a root of the equation $\sin(x) = 1 - x$ on the interval $(0, 2)$.

$f(x) = \sin(x) - 1 + x$

$f(0) = \sin(0) - 1 + 0 = -1$

$f(2) = \sin(2) - 1 + 2 = 1.9093$

Since $-1 < 0 < 1.9093$ and $f(x)$ is continuous (being a combination of trig and polynomials), there is $c \in (0,2)$ such that $f(c) = 0$

3. Find $\lim_{x\to\pi^+} x\cot(x)$.

π, positive

-1, negative

$$\lim_{x\to\pi^+} x\cot x = \lim_{x\to\pi^+} \frac{x\cdot\cos x}{\sin x}$$

$$\frac{\text{pos}\cdot\text{neg}}{\text{neg}} = \text{pos}$$

0, negative

Since the limit of the numerator is $-\pi$ and the limit of the denominator is 0, the limit does not exist. Since the function is always positive, we say $\lim_{x\to\pi^+} x\cot(x)$ DNE$(+\infty)$.

4. Find $\lim_{x\to\infty}(\sqrt{4x^2+1}-2x)$.

$$\lim_{x\to\infty}\left(\sqrt{4x^2+1}-2x\right) = \lim_{x\to\infty}\frac{\left(\sqrt{4x^2+1}-2x\right)\left(\sqrt{4x^2+1}+2x\right)}{\sqrt{4x^2+1}+2x}$$

$$= \lim_{x\to\infty}\frac{\sqrt{4x^2+1}^{\,2}+2x\sqrt{4x^2+1}-2x\sqrt{4x^2+1}-2x(2x)}{\sqrt{4x^2+1}+2x}$$

$$= \lim_{x\to\infty}\frac{4x^2+1-4x^2}{\sqrt{4x^2+1}+2x} = \lim_{x\to\infty}\frac{1}{\sqrt{4x^2+1}+2x} = 0$$