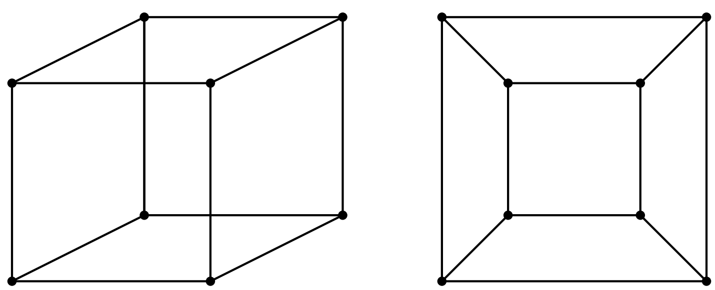
Chapter 16

Graph Theory and Polyhedra

16.1 Introduction and Motivation

If asked to draw a cube, how might you do it? You might produce one of the drawings shown in Figure 16.1



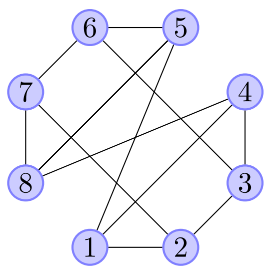
(a) (b)

Figure 16.1

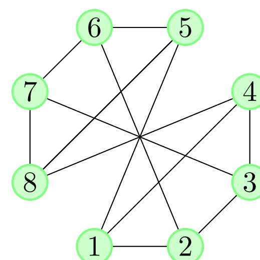
Figure 16.1(a) is a typical perspective drawing, while Figure 16.1(b) is drawn "face on." The former has the advantage that it looks somewhat like a real cube, while the latter has the feature that no line of the drawing crosses any of the others.

Yet we recognize both as cubes. Each includes the eight vertices of a cube as well as the twelve edges. Moreover, three edges are incident at each vertex in the drawings.

These properties of the drawings of a cube are not sufficient, however, to describe the adjacency of the vertices of a cube. Each of the graphs in Figure 16.2 has eight vertices and twelve edges, with three edges meeting at each vertex. Yet one can be "untangled" so that the graph of a cube is obtained, and one cannot. Can the reader decide which one yields a cube?



(a)



(b)

Figure 16.2

This suggests the following question: besides having eight vertices and twelve edges, with three edges incident at each vertex, what other properties must a graph possess so that it corresponds exactly to the adjacency of vertices on a cube?

Such questions belong to a fascinating branch of mathematics known as **graph theory.** While some mathematicians devote their entire professional lives to its study, we can provide only the barest introduction here.

In studying polyhedra from a graph-theoretical perspective, we restrict our attention to one particular aspect of polyhedra; namely, the adjacency of their vertices. Features such as the size and shape of faces are not relevant.

For example, the graph of vertex adjacency for a rectangular prism is *also* given in Figure 16.1. Since vertex adjacency is the only relevant feature under discussion, edge lengths on the polyhedron are irrelevant. As a result, Figure 16.1 represents the vertex adjacency for a parallelepiped or the frustum of a square pyramid.

Like any other branch of mathematics, graph theory has its own "language" of definitions and concepts which, over time, have been relevant to its particular application. Before going further, we must significantly enlarge our graph-theoretical vocabulary.

16.2 Basic Definitions

A graph G is essentially a drawing of points, or **vertices,** connected by lines, or **edges.** Usually a carefully drawn picture is sufficient to represent a graph. But as seen in Figure 16.1, there may be more than one way to "draw" what is essentially the same graph. To avoid such bias, it is possible to define a graph abstractly *without* a drawing by specifying a set $\mathcal{V}(G)$ of its vertices, a set $\mathcal{E}(G)$ of its edges, and a function ψ, called the **incidence function,** which specifies the ends of any particular edge. We denote the number of vertices and edges on a graph by V and E, respectively, to be consistent with the notations introduced in Chapter 2.

For example, consider a graph with $V = 6$ and $E = 12$ given by

$$\begin{gathered}\mathcal{V}(G) = \{1,2,3,4,5,6\}, \quad \mathcal{E}(G) = \{e_1, e_2, e_3, ..., e_{12}\} \\ \psi(e_1) = 12, \quad \psi(e_2) = 23, \quad \psi(e_3) = 31, \quad \psi(e_4) = 45, \\ \psi(e_5) = 56, \quad \psi(e_6) = 64, \quad \psi(e_7) = 14, \quad \psi(e_8) = 42, \\ \psi(e_9) = 25, \quad \psi(e_{10}) = 53, \quad \psi(e_{11}) = 36, \quad \psi(e_{12}) = 61.\end{gathered}$$

There are many ways to draw this graph; three are given below.

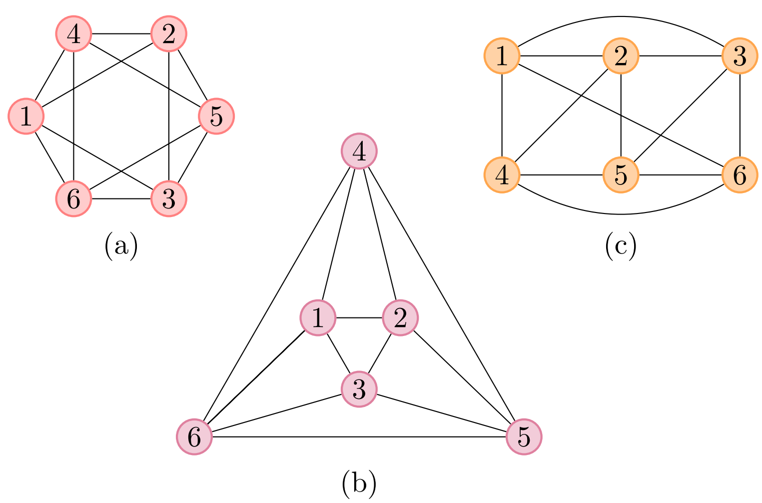


Figure 16.3

Is there a reason to prefer one of the drawings to the others? To emphasize that this graph is **planar;** that is, it can be drawn on a sheet of paper (a plane) without any of its edges crossing, one might draw the graph as in

Figure 16.3(b). To emphasize that this graph is **regular;** that is, the same number of edges are incident at each vertex, one might draw the graph as in Figure 16.3(a).

The point to be made is that a drawing of a graph is only a convenient representation. When there is ambiguity, or perhaps when necessary for proving a complex theorem, a graph can be specified in terms of $\mathcal{V}(G)$, $\mathcal{E}(G)$, and ψ as described above. For our purposes, it will usually suffice to just give a drawing of a graph.

Note that the definition of a graph does not preclude two vertices being joined by more than one edge, as in Figure 16.4.

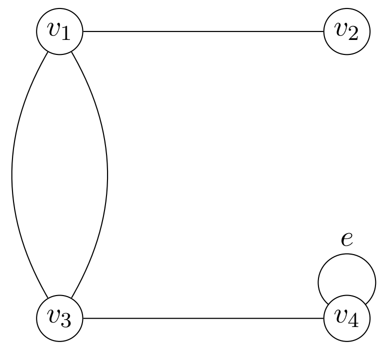


Figure 16.4

Here, two distinct edges join vertices v_1 and v_3. Also, an edge may begin and end at the same vertex, such as edge e in Figure 16.4. Such an edge is called a **loop.** Graphs with these features do not usually arise when representing adjacency of vertices on a polyhedron. When a graph has no loops and no two vertices are joined by more than one edge, the graph is said to be **simple.** Most of our discussion will focus on simple graphs.

The **degree** of a vertex $v \in \mathcal{V}(G)$, denoted by $d(v)$, is the number of edge ends incident at v. In the graph shown in Figure 16.4, $d(v_1) = 3$, and $d(v_4) = 3$ as well since the loop e contributes two ends to v_4.

A basic theorem results from counting edge ends. On the one hand, we may create the sum

$$\sum_{v \in \mathcal{V}(G)} d(v),$$

which indicates we sum the degrees of all the vertices of G. But since each edge contributes two ends to this sum, the result must be $2E$, so that

$$\sum_{v \in \mathcal{V}(G)} d(v) = 2E \tag{16.1}$$

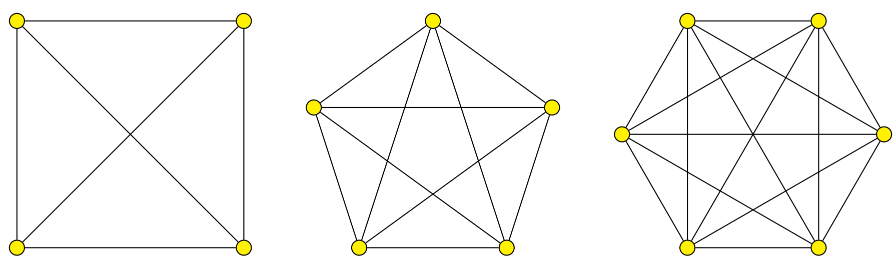
for any graph G.

It may happen, as does rather frequently with graphs derived from polyhedra, that every vertex has the same degree. Graphs with this property are called **regular graphs.** If q is the common degree of the vertices of a regular graph, the graph is said to be q–**regular.** In this case, $d(v) = q$ for all $v \in \mathcal{V}(G)$, so that (16.1) becomes

$$qV = 2E. \tag{16.2}$$

Compare this equation with (P_3') of §**2.3**.

A **complete graph** is a simple graph where each vertex is connected to every other. A complete graph with n vertices is denoted by K_n (see Figure 16.5). Note that K_n is $(n-1)$–regular.



(a) (b) (c)

Figure 16.5

A **bipartite graph** is a graph whose vertices may be partitioned into two sets, $\mathcal{V}_1(G)$ and $\mathcal{V}_2(G)$, such that each edge has one end in $\mathcal{V}_1(G)$ and the other in $\mathcal{V}_2(G)$. Such a graph is usually drawn so that the partition of vertices is clear, as in Figure 16.6.

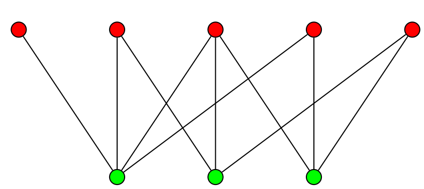


Figure 16.6

While this may seem like a lot of new definitions and concepts, they are all relevant to graphs derived from polyhedra. In the next section, we'll see how they apply to graphs derived from the Platonic solids.

16.3 The Platonic Graphs

—The Tetrahedron

As promised, we now undertake a survey of the graphs of the vertex adjacencies of the Platonic solids. The simplest is the tetrahedron, the only Platonic solid with the property that each vertex is adjacent to each of the other vertices. But this means that its graph is just the complete graph on four vertices, shown in Figure 16.5(a).

—The Cube and Octahedron

As we have seen earlier, the graph of a cube is a 3–regular simple graph with eight vertices. However, we have also seen (see Figure 16.2) that not every 3–regular, simple graph with eight vertices is the graph of a cube. So is there a way to distinguish which such graph corresponds to a cube?

An affirmative answer to this question lies with the observation that the graph of a cube is bipartite. We may see this in several ways. First, examine the graph in Figure 16.1(b) and find $\mathcal{V}_1(G)$ and $\mathcal{V}_2(G)$ as required. (The reader is encouraged to do so now.) Second, as suggested by Exercise 6(b) of Chapter 2, a regular tetrahedron may be inscribed in a cube in two ways so that the edges of the tetrahedron are diagonals of the faces of the cube. One then observes that the ends of each edge of the cube are vertices of different tetrahedra, so that the required partition of vertices may be produced by taking the vertices of the two different tetrahedra.

Finally, we may look at the coordinates for the vertices of a cube as desribed in §**14.2**. Call a vertex **even** if the number of coordinates which are -1 is even, such as $(-1, 1, -1)$ or $(1, 1, 1)$. Analogously, call a vertex **odd** if the number of coordinates which are -1 is odd, such as $(1, 1, -1)$ or $(-1, -1, -1)$. Then observe that each edge of the cube joins an even vertex and an odd vertex. Thus, the set of even vertices and the set of odd vertices form the required partition of vertices.

This is now enough. If a graph with eight vertices is simple, 3–regular, and bipartite, then it is in fact a graph of the vertex adjacency of a cube. In essence, then, there is only one such graph, unique up to a relabelling of vertices and edges.

A proof of this assertion follows. It is included as it is not difficult and illustrates how many of the concepts defined earlier may be used in combination.

> **Theorem:** *There is essentially one simple, 3–regular, bipartite graph with eight vertices. It is the graph describing the vertex adjacency of the cube.*

To see this, note that the graph must have 12 edges. This follows from

$$2E = \sum_{v \in \mathcal{V}(G)} d(v) = 8 \cdot 3,$$

since each of the 8 vertices has degree 3.

Since the graph is 3–regular, bipartite, and has 12 edges, the vertices must be partitioned into two groups of four so that a total of $12 = 4 \cdot 3$ edges end in each partition of vertices. Thus our graph looks like Figure 16.7, where one vertex is arbitrarily joined to three of the others.

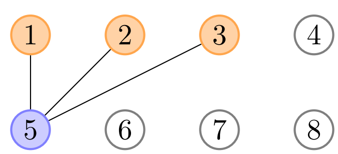


Figure 16.7

Due to the symmetry of the graph drawn thus far, there are two alternatives for vertex 6: either join 6 to vertices 1, 2, and 3, or join 6 to vertex 4 and two of the vertices 1, 2, and 3. These alternatives are drawn in Figure 16.8.

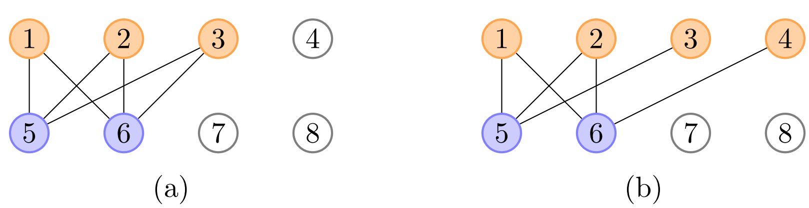
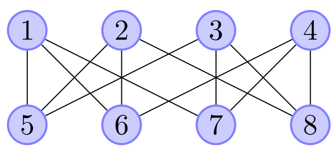


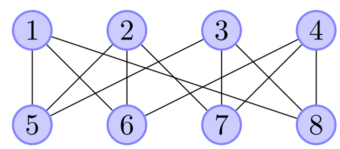
Figure 16.8

We now observe that it is impossible for Figure 16.8(a) to occur. For since the graph is 3–regular and bipartite, vertex 4 must be joined to three of the vertices 5, 6, 7, and 8. Since the graph is simple, these three vertices must all be different. But vertices 5 and 6 are already ends of three edges, leaving only two vertices, 7 and 8, available. Thus, there is no way to complete the graph drawn in Figure 16.8(a).

Now since vertices 5 and 6 are already ends of three edges, vertices 3 and 4 must each be joined to vertices 7 and 8. This leaves just two ways to consistently complete the graph in Figure 16.8(b): either join vertices 1 and 7, and 2 and 8, or join vertices 1 and 8, and 2 and 7. Both alternatives are shown in Figure 16.9.



(a)



(b)

Figure 16.9

Now redraw Figure 16.9(b) by switching the locations of vertices 7 and 8, but maintaining adjacencies of the vertices. The result is Figure 16.10.

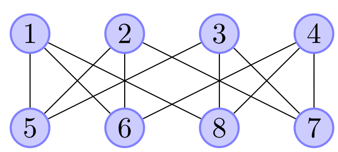


Figure 16.10

But Figure 16.10 looks exactly like Figure 16.9(a)! This means that Figures 16.9(a) and (b) are essentially the *same* graph, since one may be obtained from the other by simply relabelling some of the vertices (in this case, switching vertices 7 and 8). Such graphs are said to be **isomorphic**.

Moreover, by redrawing Figure 16.9(a); that is, by putting its vertices at different locations on paper while maintaining vertex adjacency, the familiar Figure 16.11 is obtained.

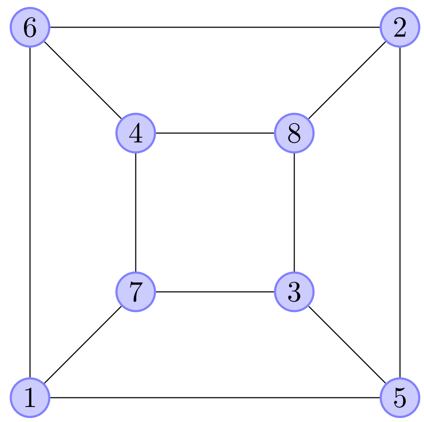


Figure 16.11

This demonstrates that if we adhere to the conditions that our graph is simple, 3–regular, bipartite, and has eight vertices, we *must* eventually create a graph such as Figure 16.11, a representation of the cube. In other words, the Theorem is proved.

–The Icosahedron and Dodecahedron

A graph of the vertex adjacency of the icosahedron must have twelve vertices, thirty edges, and be 5–regular since five edges meet at each vertex of the icosahedron. There are many graphs with these properties, so there is no simple theorem analogous to our recent result about the cube. This graph is shown in Figure 16.12.

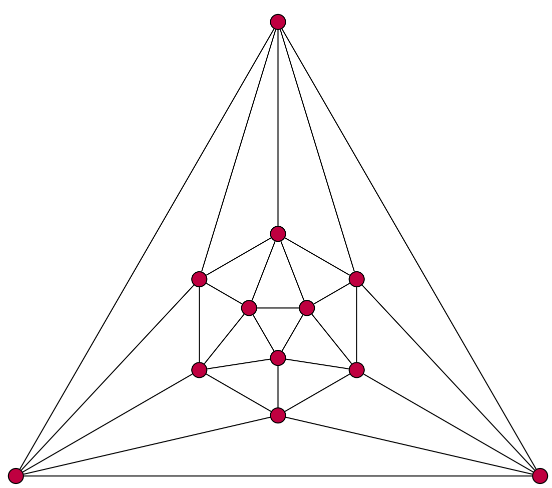


Figure 16.12

Likewise, there is no simple theorem characterizing the graph of vertex adjacency of the dodecahedron. It must, however, have 20 vertices, 30 edges, and be 3–regular. This graph is shown in Figure 16.13.

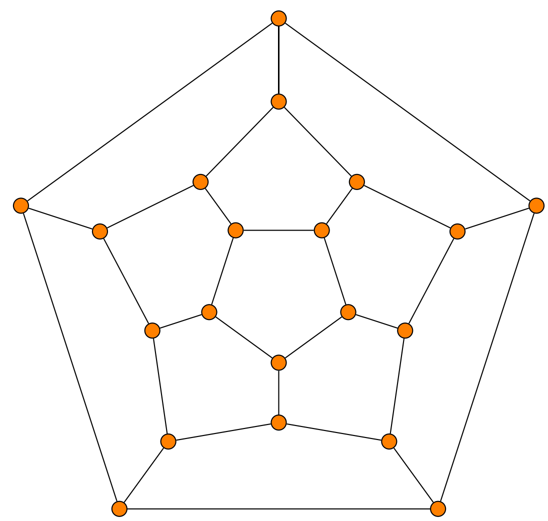


Figure 16.13

16.4 Exercises

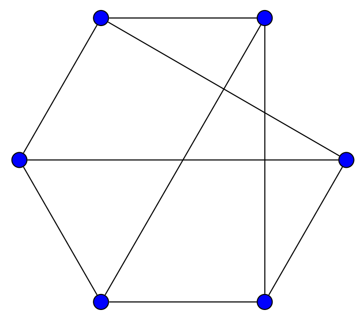
1. Show that there is no 3–regular graph with seven vertices.

2. Decide for which positive integers q and V it is possible to create a q–regular graph on V vertices. It may be helpful to think in terms of an algorithm for creating such a graph.

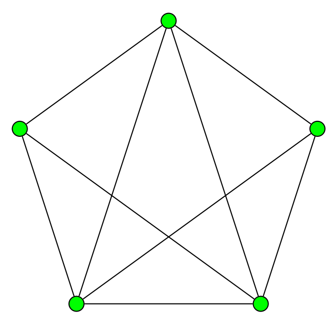
   When is it possible to create a *simple,* q–regular graph on V vertices? A simple, bipartite, q–regular graph on V vertices? Again, thinking in terms of an algorithm may be fruitful.

3. The graphs below show the vertex adjacency for various convex polyhedra. Redraw the graphs so that the polyhedra are readily apparent. For, example, Figure 16.2 may be redrawn to yield a cube as shown in Figure 16.1. (Hint: It may be helpful to use Euler's formula to determine the number of faces on the polyhedron.)

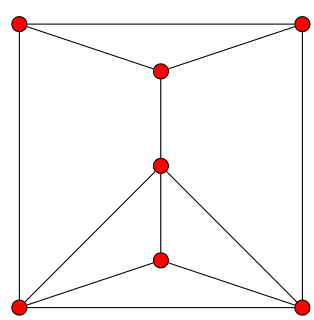
(a)



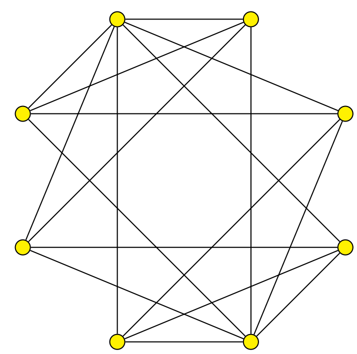
(b)



(c)



(d)



4. Prove the following statement, using logic similar to that used in proving the theorem in §**16.3**: There is essentially one 4–regular simple graph on six vertices: it is the graph of the vertex adjacency of the octahedron.

5. (a) Show that the graph of the vertex adjacency of a hexagonal prism is simple, 3–regular, and bipartite.
   (b) By giving a counterexample, show that a simple, 3–regular bipartitite graph with twelve vertices need not necessarily correspond to the graph of the vertex adjacency of a hexagonal prism.

6. A graph is said to have a **perfect matching** if there is a set of edges $\mathcal{M}$ with the following properties:
   (a) no two edges in $\mathcal{M}$ are incident; that is, no two edges in $\mathcal{M}$ have a vertex in common, and
   (b) every vertex of the graph is the end of some edge in $\mathcal{M}$.

   For each of the Platonic graphs, find a perfect matching.

7. A graph is said to be **tripartite** if the vertices may be partitioned into three sets $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_3$ such that the ends of each edge lie in two different sets, and any pair of sets contains the ends of at least one edge.
   (a) Find a simple, 5–regular, tripartite graph with twelve vertices.
   (b) Show that the graph of the vertex adjacency of the icosahedron (see Figure 16.12) is not tripartite.