The Inverse Chain Rule

There are many techniques for finding antiderivatives. The first one is called using **substitution,** which we'll call the Inverse Chain Rule.

Let's review the Chain Rule by looking at $\frac{d}{dx}\sin(x^2)$. We use $f(x) = \sin(x)$ and $g(x) = x^2$, so that $f'(x) = \cos(x)$ and $g'(x) = 2x$. So

$$\begin{aligned}\frac{d}{dx}\sin(x^2) &= f'(g(x))g'(x) \\ &= \cos(g(x)) \cdot 2x \\ &= 2x\cos(x^2).\end{aligned}$$

Using antiderivative notation, we would write

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

We can generalize to any function composition by writing

$$\int f'(g(x))g'(x)\,dx = f(g(x)) + C. \tag{1}$$

How would we use this if we were just given

$$\int 2x\cos(x^2)\,dx\ ? \tag{2}$$

We look at the **integrand,** which is the function to be integrated. Thus, $f'(g(x))g'(x)$ is the integrand in (1), and $2x\cos(x^2)$ is the integrand in (2).

Notice in (1) that you see a $g(x)$ and a $g'(x)$ in the integrand. That's our starting point. Looking at $2x\cos(x^2)$, can you see a $g(x)$ and a $g'(x)$? Yes, you've got x^2 and $2x$, and $2x$ is the derivative of x^2. So we know that $g(x) = x^2$.

How do we use this information? We make what is called a **substitution,** which is just $g(x)$. And just like with the Fundamental Theorem of Calculus, we need a different letter, which is usually u. In other words, we substitute $u = x^2$. You can see why we don't use x again, since writing $x = x^2$ is very confusing.

So $u = x^2$ and $\frac{du}{dx} = 2x$. Let's rewrite using this substitution.

$$\begin{aligned}\int 2x\cos(x^2)\,dx &= \int \cos(x^2)\cdot 2x\,dx \\ &= \int \cos(u)\cdot\frac{du}{dx}\cdot dx \\ &= \int \cos(u)\,du.\end{aligned}$$

It might look a little odd to cancel out the dx's. But this is an example of why the notation $\frac{du}{dx}$ is sometimes used instead of $u'(x)$. It makes the substitution process much easier.

Now the integral has become a lot simpler. In fact, this is one of the basic antiderivatives. So

$$\begin{aligned}\int \cos(u)\,du &= \sin(u) + C \\ &= \sin(x^2) + C. \qquad \text{substituting back, since } u = x^2\end{aligned}$$

Example 1

Find $\displaystyle\int (1-2x)e^{x-x^2}\,dx$.

We first look at the integrand and see if can spot a $g(x)$ and $g'(x)$. Yes, $g(x) = x - x^2$ works, so we use the substitution $u = x - x^2$, so $\dfrac{du}{dx} = 1 - 2x$. Now rewrite.

$$\begin{aligned}\int (1-2x)e^{x-x^2}\,dx &= \int e^{x-x^2}(1-2x)\,dx \\ &= \int e^u \cdot \frac{du}{dx} \cdot dx \\ &= \int e^u\,du \\ &= e^u + C \\ &= e^{x-x^2} + C.\end{aligned}$$

You should notice that done correctly, substituting will eliminate *all* the x's, so the only variable will be u. If this does not happen, then you need to try another substitution.

There is another way to write out the substitution, which you will find in most other resources. It's the same algebra, though done in a slightly different order.

We start the same way: substitute $u = x - x^2$, so that $\dfrac{du}{dx} = 1 - 2x$. Now solve for du.

$$\begin{aligned}\frac{du}{dx} &= 1 - 2x \\ \frac{du}{dx} \cdot dx &= (1-2x)\,dx \\ du &= (1-2x)\,dx\end{aligned}$$

So when we rewrite, we get

$$\begin{aligned}\int (1-2x)e^{x-x^2}\,dx &= \int e^{x-x^2}(1-2x)\,dx \\ &= \int e^u\,du \\ &= e^u + C \\ &= e^{x-x^2} + C.\end{aligned}$$

Basically, we canceled out the dx's first by solving for du. We'll stick to this method, as it is more commonly used.

Example 2

Find $\int \sin(3x)\,dx$.

At first glance, it looks like we can't find both $g(x)$ and $g'(x)$. We might think $g(x) = 3x$ would work, but $g'(x) = 3$, and we don't see a 3.

But since 3 is a constant, we can multiply and divide by 3 as follows.

$$\begin{aligned}\int \sin(3x)\,dx &= \int \frac{1}{3}\cdot 3\sin(3x)\,dx\\ &= \frac{1}{3}\int 3\sin(3x)\,dx.\end{aligned}$$

Remember, we can factor constants out of derivatives and integrals, so the last step is legitimate. Now we have $g'(x)$, so we can make the substitution $u = 3x$, so that $\frac{du}{dx} = 3$ and $du = 3\,dx$. Rewriting, we get

$$\begin{aligned}\frac{1}{3}\int 3\sin(3x)\,dx &= \frac{1}{3}\int \sin(3x)\cdot 3\,dx\\ &= \frac{1}{3}\int \sin(u)\,du\\ &= \frac{1}{3}(-\cos(u)) + C\\ &= -\frac{1}{3}\cos(3x) + C\end{aligned}$$

IMPORTANT!!!!

This can **ONLY** be done because 3 is a number. You cannot do this otherwise.

$$\begin{aligned}\int \cos(x^2)\,dx &= \int \frac{1}{2x}\cdot 2x\cos(x^2)\,dx\\ &= \frac{1}{2x}\int 2x\cos(x^2)\,dx. \qquad \textbf{DON'T DO THIS!!!}\end{aligned}$$

So be careful. Only use this trick if your derivative is off by a **CONSTANT MULTIPLE.**

We summarize the steps below.

Inverse Chain Rule

To integrate $\int f'(g(x))g'(x)\,dx$:

1. Look for a $g(x)$ and $g'(x)$ pair in the integrand – $g'(x)$ can be off by a constant multiple;
2. If $g'(x)$ is off by a constant multiple, multiply and divide by this constant and factor out;
3. Substitute $u = g(x)$, and solve for $du = g'(x)\,dx$;
4. Rewrite the integral in terms of u; all x's should disappear;
5. Find the antiderivative with respect to u;
6. Substitute back to rewrite in terms of x only.

You'll only get better at substitution by practicing. The main trick is to spot $g(x)$ and $g'(x)$. Once you do this, just follow the steps one at a time. Be sure to have your table of antiderivatives handy.

Example 3

Find $\int x^3(x^4+2)^5\,dx$.

Let's go one step at a time.

1. We know that the derivative of x^4+2 is $4x^3$, which is good, since we're only off by a constant multiple of 4.

2. We now rewrite:

$$\int \frac{1}{4}\cdot 4\cdot x^3(x^4+2)^5\,dx = \frac{1}{4}\int 4x^3(x^4+2)^5\,dx.$$

3. We now substitute $u = x^4+2$ so that $\dfrac{du}{dx} = 4x^3$ and $du = 4x^3\,dx$.

4. Rewrite again:

$$\begin{aligned}\frac{1}{4}\int 4x^3(x^4+2)^5\,dx &= \frac{1}{4}\int (x^4+2)^5\cdot 4x^3\,dx\\ &= \frac{1}{4}\int u^5\,du\end{aligned}$$

5. Now take the antiderivative.

$$\begin{aligned}\frac{1}{4}\int u^5\,du &= \frac{1}{4}\left(\frac{1}{6}u^6\right)+C\\ &= \frac{1}{24}u^6+C.\end{aligned}$$

6. Finally, substitute back.

$$\begin{aligned}\int x^3(x^4+2)^5\,dx &= \frac{1}{24}u^6+C\\ &= \frac{1}{24}(x^4+2)^6+C.\end{aligned}$$

Example 4

Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

1. Because the derivative of $\ln x$ is $\dfrac{1}{x}$, we choose $g(x) = \ln x$.
2. We've got it exactly, no need to adjust.
3. Now substitute $u = \ln x$, so that $du = \dfrac{1}{x}\,dx$.
4. Rewrite.
$$\int \ln x \cdot \frac{1}{x}\,dx = \int u\,du.$$
5. Taking the antiderivative, we have
$$\int u\,du = \frac{1}{2}u^2 + C.$$
6. Substituting back, we get
$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \frac{1}{2}u^2 + C \\ &= \frac{1}{2}(\ln x)^2 + C.\end{aligned}$$

Example 5

Find $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\,dx.$

This problem is similar to Example 2. We notice that it looks pretty close to the derivative of $\arcsin(x)$, except for the factor of 4.

1. Let's see what happens if we make $g(x) = 2x$. Then $g'(x) = 2$. Remember, we can be off by a constant multiple.

2. So we can rewrite as
$$\frac{1}{2}\int \frac{2}{\sqrt{1-4x^2}}\,dx.$$

3. Now substitute $u = 2x$, so that $4x^2 = (2x)^2 = u^2$ and $du = 2\,dx$.

4. Rewriting again, we have
$$\begin{aligned}\frac{1}{2}\int \frac{2}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\int \frac{1}{\sqrt{1-(2x)^2}}\cdot 2\,dx\\ &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du\end{aligned}$$

   Can you see why using $g(x) = 2x$ was a good idea? To get an $\arcsin(x)$ in our answer, the derivative has to match *exactly.* In a sense, the substitution $u = 2x$ allows us to get rid of the factor of 4. This technique is useful for the derivatives of inverse trigonometric functions.

5. Now we can take the antiderivative.
$$\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin(u) + C.$$

6. Substituting back, we get
$$\begin{aligned}\int \frac{1}{\sqrt{1-4x^2}}\,dx &= \frac{1}{2}\arcsin(u) + C\\ &= \frac{1}{2}\arcsin(2x) + C.\end{aligned}$$

Homework

1. Find $\int x^5(1-x^6)^3\,dx$.
2. Find $\int 3x^2\sin(x^3)\,dx$.
3. Find $\int e^{5x}\,dx$.
4. Find $\int \frac{1}{1+9x^2}\,dx$.
5. Find $\int \frac{(\ln x)^2}{3x}\,dx$.
6. Find $\int x\,3^{x^2}\,dx$.
7. Find $\int e^x\cos(e^x)\,dx$.
8. Find $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.

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1. Use $u = 1 - x^6$, so that $\dfrac{du}{dx} = -6x^5$ and $du = -6x^5\,dx$. We're off by a factor of -6, so we compensate and then substitute.

$$\begin{aligned}
\int x^5(1-x^6)^3\,dx &= \frac{1}{-6}\int -6x^5(1-x^6)^3\,dx \\
&= -\frac{1}{6}\int (1-x^6)^3(-6x^5\,dx) \\
&= -\frac{1}{6}\int u^3\,du \\
&= -\frac{1}{6}\cdot\frac{1}{4}u^4 + C \\
&= -\frac{1}{24}(1-x^6)^4 + C.
\end{aligned}$$

2. Use $u = x^3$, so that $\dfrac{du}{dx} = 3x^2$ and $du = 3x^2\,dx$. It's an exact match, so no need to compensate.

$$\begin{aligned}
\int 3x^2\sin(x^3)\,dx &= \int \sin(x^3)\cdot 3x^2\,dx \\
&= \int \sin(u)\,du \\
&= -\cos(u) + C \\
&= -\cos(x^3) + C.
\end{aligned}$$

3. We use $u = 5x$ so that $\dfrac{du}{dx} = 5$ and $du = 5\,dx$. We're off by a factor of 5, we we'll need to compensate.

$$\begin{aligned}
\int e^{5x}\,dx &= \frac{1}{5}\int 5e^{5x}\,dx \\
&= \frac{1}{5}\int e^{5x}\cdot 5\,dx \\
&= \frac{1}{5}\int e^{u}\,du \\
&= \frac{1}{5}e^{u} + C \\
&= \frac{1}{5}e^{5x} + C.
\end{aligned}$$

4. This looks like an $\arctan(x)$ will be involved. We rewrite as $\int \frac{1}{1+(3x)^2}\,dx$, and so we use $u = 3x$. Then $\frac{du}{dx} = 3$ and $du = 3\,dx$. We're off by a factor of 3.

$$\begin{aligned}
\int \frac{1}{1+9x^2}\,dx &= \frac{1}{3}\int 3 \cdot \frac{1}{1+(3x)^2}\,dx \\
&= \frac{1}{3}\int \frac{1}{1+(3x)^2} \cdot 3\,dx \\
&= \frac{1}{3}\int \frac{1}{1+u^2}\,du \\
&= \frac{1}{3}\arctan(u) + C \\
&= \frac{1}{3}\arctan(3x) + C.
\end{aligned}$$

5. We use $u = \ln x$, so that $\frac{du}{dx} = \frac{1}{x}$ and $du = \frac{1}{x}\,dx$. We don't need the 3, so we just factor it out.

$$\begin{aligned}
\int \frac{(\ln x)^2}{3x}\,dx &= \frac{1}{3}\int (\ln x)^2 \cdot \frac{1}{x}\,dx \\
&= \frac{1}{3}\int u^2\,du \\
&= \frac{1}{3} \cdot \frac{1}{3}u^3 + C \\
&= \frac{1}{9}(\ln x)^3 + C.
\end{aligned}$$

6. We use $u = x^2$, so that $\frac{du}{dx} = 2x$ and $du = 2\,dx$. We're off by a factor of 2.

$$\begin{aligned}
\int x\,3^{x^2}\,dx &= \frac{1}{2}\int 2x\,3^{x^2}\,dx \\
&= \frac{1}{2}\int 3^{x^2} \cdot 2x\,dx \\
&= \frac{1}{2}\int 3^u\,du \\
&= \frac{1}{2} \cdot \frac{3^u}{\ln 3} + C \\
&= \frac{3^{x^2}}{2\ln 3} + C.
\end{aligned}$$

7. We use $u = e^x$, so that $\dfrac{du}{dx} = e^x$ and $du = e^x\,dx$. We've got an exact match.

$$\begin{aligned}\int e^x \cos(e^x)\,dx &= \int \cos(e^x)\cdot e^x\,dx \\ &= \int \cos(u)\,du \\ &= \sin(u) + C \\ &= \sin(e^x) + C.\end{aligned}$$

8. In the denominator, we see an expression which looks like an $\arcsin(x)$ is involved. Rerwriting as $\displaystyle\int \frac{e^x}{\sqrt{1-(e^x)^2}}\,dx$, we use the substitution $u = e^x$. Then $\dfrac{du}{dx} = e^x$ and $du = e^x\,dx$, which gives us an exact match.

$$\begin{aligned}\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx &= \int \frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x\,dx \\ &= \int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \arcsin(u) + C \\ &= \arcsin(e^x) + C.\end{aligned}$$