The Area Function

Closely related to areas are **area functions.** Let's begin with an example. We see below in Figure 1 part of the graph of $f(x) = \dfrac{1}{x}$.

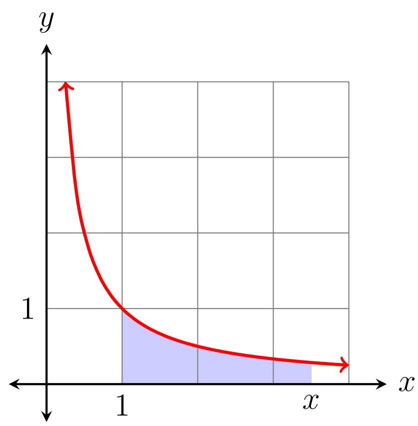

Figure 1: An area function.

First you need a starting point, which is $x_0 = 1$ in this case. You can use any value in the domain of the function. Often $x_0 = 0$ is easy to use, but our function is not defined there. Then you need a function, which is $f(x) = \dfrac{1}{x}$ in our case. With these you can define the **area function**

$$A(x) = \int_{x_0}^{x} f(u)\,du,$$

which in our case is

$$A(x) = \int_{1}^{x} \frac{1}{u}\,du.$$

What this means is that $A(x)$ gives you the area under a curve $f(x)$ and above the x-axis, measured starting from the point $x = x_0$. We use the variable u since it would be confusing to write

$$A(x) = \int_{x_0}^{x} f(x)\,dx,$$

because on one hand we're using x to represent a number on the x-axis, and on the other hand, we're using x as a variable in a function.

Let's take a moment to see why we can do this. Let's compute $\displaystyle\int_0^2 2u\,du$. Remember, the "$du$" means we're using the variable u in our functions. Then u^2 is an antiderivative for $2u$, and so

$$\int_0^2 2u\,du = 2^2 - 0^2 = 4.$$

But if we used a different variable, say "x," then an antiderivative of $2x$ would be x^2, and so

$$\int_0^2 2x\,dx = 2^2 - 0^2 = 4.$$

We get the same answer either way, since we're always plugging back into a function and evaluating. However, notice that

$$\int 2u\,du = u^2 + C, \qquad \int 2x\,dx = x^2 + C.$$

So when evaluating an *indefinite* integral, your answer will be in terms of whatever variable you're using. But when evaluating a definite integral, as we just saw, you'll always get the same answer no matter what variable you use.

So using "u" in

$$A(x) = \int_1^x \frac{1}{u}\,du$$

won't affect our answer. So let's work this out, remembering that $\ln u$ is an antiderivative of $1/u$.

$$\begin{aligned} A(x) &= \int_1^x \frac{1}{u}\,du \\ &= \ln x - \ln 1 \\ &= \ln x. \end{aligned}$$

The important observation here is that

$$A'(x) = \frac{1}{x} = f(x).$$

Notice that since $a = 1$ and $b = x$, our final answer is in terms of x.

But what if we used a *different* x_0 instead? Like $x_0 = 2$? Let's see.

$$\begin{aligned} A(x) &= \int_2^x \frac{1}{u}, du \\ &= \ln x - \ln 2. \end{aligned}$$

So we get a slightly different area function, since we're starting to measure area from a different place. But we *still* have

$$A'(x) = \frac{1}{x} = f(x).$$

We summarize this as follows.

Fundamental Theorem of Calculus, Part II

Let $f(x)$ be a continuous function, and let x_0 be a point in the domain of $f(x)$. If the area function $A(x)$ is defined by

$$A(x) = \int_{x_0}^{x} f(u)\, du,$$

then

$$A'(x) = f(x).$$

Again, we emphasize that $A(x)$ will depend upon what you choose for x_0, but whatever you choose, you will *always* get $A'(x) = f(x)$. We do need to assume that $f(x)$ is continuous, but this will be the case with all of our examples. We only need to include it for mathematical correctness, and you will notice this assumption if you look at other resources.

Often, the Fundamental Theorem of Calculus is stated without saying anything about area functions. Instead, it is stated as:

$$\frac{d}{dx} \int_{x_0}^{x} f(u)\, du = f(x).$$

This is equivalent to the way it is stated above, but you should be aware if you look at other resources.

Combined with the Fundamental Theorem of Calculus, Part I, we have essentially this. If you have a function $f(x)$, you antidifferentiate to get the corresponding area function. And if you are given an area function $A(x)$, you take the derivative to see what function it corresponds to. This is another way to say that differentiation and antidifferentiation are inverse processes.

Another way to say it is this. Suppose I give you a function, $f(x)$. I ask you two questions. First, find an antiderivative for $f(x)$. Next, write a function which calculates the area under the graph of $f(x)$ from a given starting point. Is there any reason to think that these two questions have the same answer? This is why the Fundamental Theorem of Calculus is so important.

It is also important to point out that we didn't actually *prove* the Fundamental Theorem of Calculus, but rather saw why it should be true through examples. That's good enough for our purposes. It is more important to understand what it means than how to prove it.

Example 1

We'll create an area function for $f(x) = x^2$ and $x_0 = -1$. The graph of $f(x) = x^2$ is shown at the left of Figure 2.

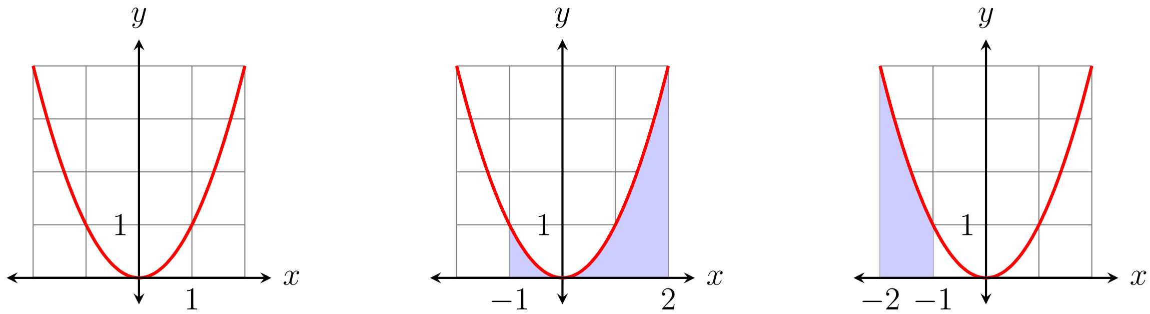

Figure 2: Areas bounded by $f(x) = x^2$.

$$\begin{aligned} A(x) &= \int_{-1}^{x} f(u)\,du \\ &= \int_{-1}^{x} u^2\,du \\ &= \frac{1}{3}u^3\Big|_{-1}^{x} && \text{New notation for evaluating.} \\ &= \frac{1}{3}(x^3 - (-1)^3) \\ &= \frac{1}{3}(x^3 + 1). \end{aligned}$$

Note the new notation for evaluating. What is this notation good for? Suppose we wanted to find the area underneath $f(x) = x^2$ but above the x-axis on the interval $[-1, 2]$. On the one hand, this is just

$$A(2) = \frac{1}{3}(2^3 + 1) = 3.$$

Area functions are useful when you are evaluating many different areas. But if you just have to evaluate *one* area, which is often the case, this new notation lets you skip the area function altogether. You can just write

$$\begin{aligned} \int_{-1}^{2} f(x)\,dx &= \int_{-1}^{2} x^2\,dx \\ &= \frac{1}{3}x^3\Big|_{-1}^{2} && \text{New notation for evaluating.} \\ &= \frac{1}{3}(2^3 - (-1)^3) \\ &= 3. \end{aligned}$$

Note that because our limits are *numbers,* we can use x as the variable. This is usually how it's done, but you can continue to use u if you want. Again, you need to be familiar with the standard notation in looking at other resources. If you need to find an *area function,* you'll need to use both x and u. But if you're just looking for an area, you can just use x. You should also be aware that many books use t instead of u, but we use t for time only.

Now let's look at the area underneath the parabola and above the x-axis on the interval $[-1, -2]$. So if we use the area function, we get that the area is

$$A(-2) = \frac{1}{3}((-2)^3 + 1) = -\frac{7}{3}.$$

But our area is *above* the x-axis, so how can it be negative? This is because we are measuring our area starting at $x = -1$, and have to go along the *negative* x-direction to get the area. Notice that

$$A(-2) = \int_{-1}^{-2} f(u)\, du.$$

Now let's evaluate (using our new notation) the area under the parabola and above the x-axis using $a = -2$ and $b = -1$:

$$\begin{aligned} \int_{-2}^{-1} f(x)\, dx &= \int_{-2}^{-1} x^2\, dx \\ &= \frac{1}{3}x^3 \bigg|_{-2}^{-1} \\ &= \frac{1}{3}((-1)^3 - (-2)^3) \\ &= \frac{1}{3}(-1 + 8) \\ &= \frac{7}{3}. \end{aligned}$$

This may look at little odd, but it is necessary. In geometry, areas were *always* positive. But in calculus, they can be negative. Because the Fundamental Theorem of Calculus is the central result relating differentiation/finding slopes to antidifferentiation/finding areas, it is *only* true if we introduce the concept of a negative area. So whether an area is positive or negative depends on two things: (1) whether the region is above or below the x-axis, or whether the region is traversed from the right or from the left. We see examples which summarize this in Figure 3.

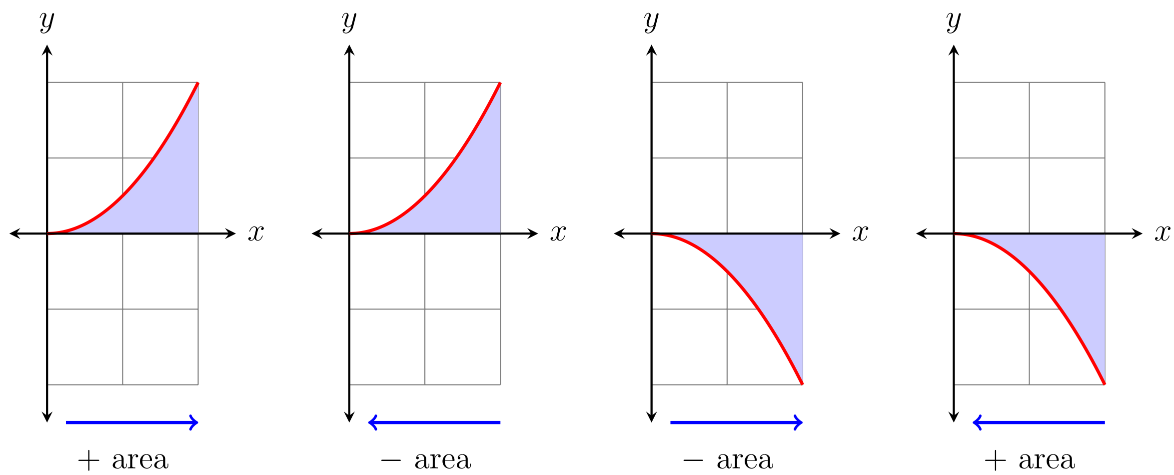

Figure 3: Above/below x-axis, traversing right/left.

Our example also showed that

$$\int_{-2}^{-1} f(x)\,dx = \frac{7}{3} = -\left(-\frac{7}{3}\right) = -\int_{-1}^{-2} f(x)\,dx.$$

This can be written in general; when a and b are in the domain of $f(x)$, then

$$\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx.$$

This is because whether the area is above or below the x-axis, one integral measures the area from left to right, and the other measures from right to left. So they must be opposites of each other.

Example 2

Write an area function $A(x)$ for $f(x) = \sin(x)$ with $x_0 = 0$.

1. Verify that $A'(x) = f(x)$.
2. Find all the values of x where the area function is equal to 0.
3. Explain, using the graph of $\sin(x)$, why this makes sense geometrically.

Solution

To find the area function, calculate

$$\begin{aligned}\int_0^x \sin(x)\, dx &= -\cos(x)\Big|_0^x \\ &= -\cos(x) - (-\cos(0)) \\ &= 1 - \cos(x)\end{aligned}$$

1. $$\frac{d}{dx}(1 - \cos(x)) = 0 - (-\sin(x)) = \sin(x).$$
2. Solving $A(x) = 0$ is the same as solving $\cos(x) = 1$. This occurs at all multiples of 2π: 0, 2π, 4π, etc., as well as -2π, -4π, etc.
3. Look at the graph of $\sin(x)$ on desmos. You will notice that starting at $x_0 = 0$ and going in either direction, every time you hit a multiple of 2π, you'll see that the positive areas *exactly* cancel out with the negative areas. Thus, the cumulative area determined by $A(x)$ must be 0.

Example 3

Suppose $f(x)$ is a function such that $\int_0^2 f(x)\,dx = -4$. What is $\int_2^0 f(x)\,dx$?

$\int_2^0 f(x)\,dx = -(-4) = 4$, since it still traces the area over the interval $[0, 2]$, but in the *opposite* direction.

Example 4

Find $\dfrac{d}{dx}\displaystyle\int_2^x \arctan(u)\,du$.

From the Fundamental Theorem of Calculus, this is just $\arctan(x)$.

Example 5

Find $\dfrac{d}{dx}\displaystyle\int_x^3 e^{2u}\,dxu$

It is very important that the integral matches the Fundamental Theorem of Calculus *exactly.* The x must be in the upper limit, not the lower. So we need to switch first, as seen below.

$$\begin{aligned}\frac{d}{dx}\int_x^3 e^{2u}\,du &= -\frac{d}{dx}\int_3^x e^{2u}\,du \\ &= -e^{2x}.\end{aligned}$$

Example 6

Consider the function $f(x)$, graphed below.

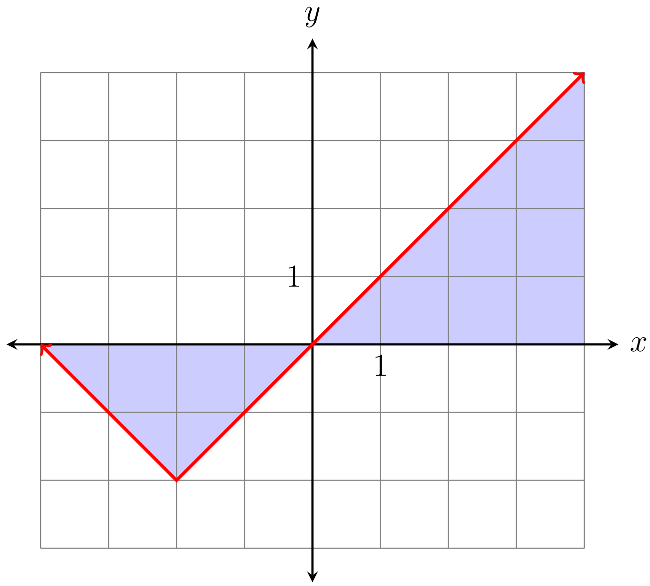

Figure 4: Evaluating areas.

Using simple geometry (no integrals needed), evaluate the following.

1. $\int_{-4}^{4} f(x)\,dx$.
2. $\int_{-2}^{2} f(x)\,dx$.
3. $\int_{4}^{0} f(x)\,dx$.
4. $\int_{3}^{-2} f(x)\,dx$.
5. $\int_{3}^{3} f(x)\,dx$.

Divide the areas into triangles and trapezoids. Remember that areas below the x-axis are negative, and traversing an interval from right to left changes the sign of the area.

Solutions

1. 4.
2. 0.
3. -8.
4. $-\frac{5}{2}$.
5. 0.

Homework

1. Find an area $A(x)$ function using $x_0 = 1$ and $f(x) = \dfrac{1}{x^4}$. Verify that $A'(x) = f(x)$.
2. Graph $f(x) = \arcsin(\sin(x))$ on desmos. Let $A(x)$ be the area function for $f(x)$ with $x_0 = 0$. Using geometry, find all values of x such that $A(x) = 0$.
3. Suppose $f(x)$ is a function such that $\displaystyle\int_2^6 f(x)\,dx = 10$. What is $\displaystyle\int_6^2 f(x)\,dx$?
4. Find $\displaystyle\frac{d}{dx}\int_3^x \ln(x^2+1)\,dx$.
5. Find $\displaystyle\frac{d}{dx}\int_x^7 \sin(3x-\pi)\,dx$.
6. Consider the function $f(x)$, graphed below.

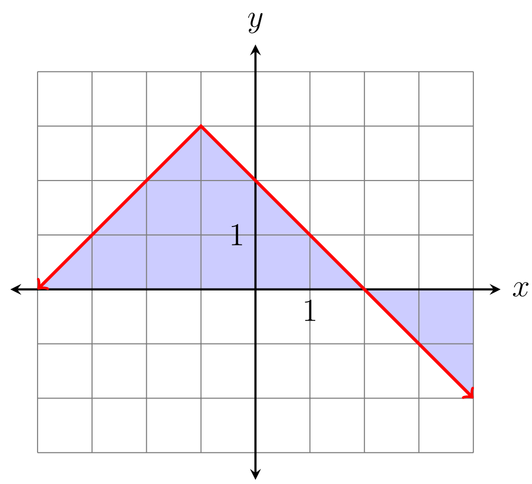

Figure 5: Evaluating areas.

Using simple geometry (no integrals needed), evaluate the following.

(a) $\displaystyle\int_{-4}^{4} f(x)\,dx$.

(b) $\displaystyle\int_{-2}^{4} f(x)\,dx$.

(c) $\displaystyle\int_{4}^{0} f(x)\,dx$.

(d) $\displaystyle\int_{2}^{-2} f(x)\,dx$.

(e) $\displaystyle\int_{1}^{1} f(x)\,dx$.

Solutions

1. $$\begin{aligned} A(x) &= \int_1^x u^{-4}\,du \\ &= -\frac{1}{3}u^{-3}\Big|_1^x \\ &= -\frac{1}{3}x^{-3} + \frac{1}{3} \end{aligned}$$

   Then
   $$A'(x) = -\frac{1}{3}(-3x^{-4}) = \frac{1}{x^4}.$$

2. This problem is very similar to Example 2. Using the same logic, we have that $A(x) = 0$ when x is a multiple of 2π, including negative multiples as well.

3. $$\int_6^2 f(x)\,dx = -\int_2^6 f(x)\,dx = -10.$$

4. Using the Fundamental Theorem of Calculus, this is just $\ln(x^2+1)$.

5. Using the Fundamental Theorem of Calculus,
   $$\frac{d}{dx}\int_x^7 \sin(3x-\pi)\,dx = -\frac{d}{dx}\int_7^x \sin(3x-\pi) = -\sin(3x-\pi).$$

6. (a) 7.
   (b) 5.
   (c) 0.
   (d) -7.
   (e) 0.