Areas

We've come around full circle at this point. We begin with Example 2 from the handout for Day 3 of the semester, illustrated in Figure 1.

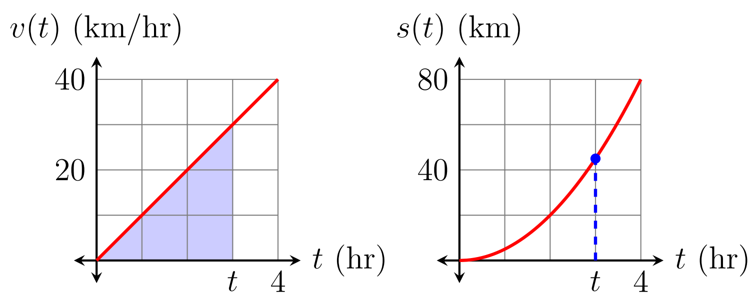
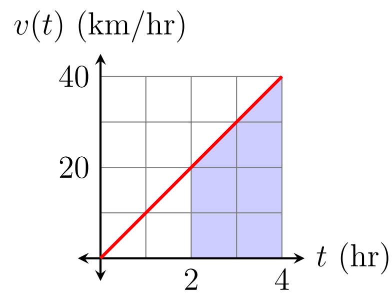


Figure 1: Velocity graph (left), and displacement graph (right).

Here, the velocity is given by $v(t) = 10t$ since it is linear, and at $t = 4$, we are at 40 km/hr. The displacement $s(t)$ up to time t is the area under the velocity curve up to time t. By looking at the areas of triangles, we found out that $s(t) = 5t^2$. If you look closely, you'll observe that $10t$ is the derivative of $5t^2$, and $5t^2$ is an antiderivative of $10t$, since

$$\begin{aligned} \int 10t\,dt &= 10\left(\frac{1}{2}t^2\right) + C \\ &= 5t^2 + C. \end{aligned}$$

Now let's ask the following question: How far have we driven between time $t = 2$ and $t = 4$? We could shade in the following area.



We could calculate this using the formula for the area of a trapezoid. But it turns out that there is a simpler way which is very important in calculus. We know that $s(t)$ measures the area under the velocity curve, but *starting at* $t = 0$. What do we do when we start at $t = 2$?

The key is to look at the blue trapezoid as the *difference* of two areas, as shown in Figure 2. Most labels have been removed to make the geometry easier to see.

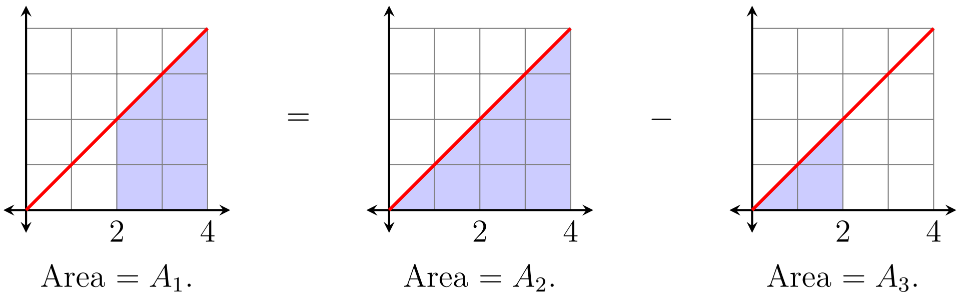


Figure 2: One area as the difference of two others.

As you can see, if you take the area of the large triangle, A_2, and subtract off the area of the small triangle, A_3, you get the area of the trapezoid, which is the distance traveled between time $t = 2$ and $t = 4$. Take a moment to really understand this from the figures.

Why is this significant? Because the areas A_2 and A_3 have their left endpoints at $t = 0$. This means that $A_2 = s(4)$, the displacement from time $t = 0$ to time $t = 4$, and $A_3 = s(2)$. But we know what $s(t)$ is, so we can compute the area A_1.

$$\begin{aligned} s(t) &= 5t^2 \\ A_1 &= A_2 - A_3 \\ &= s(4) - s(2) \\ &= 5(4^2) - 5(2^2) \\ &= 80 - 20 \\ &= 60. \end{aligned}$$

Of course there is nothing special about $t = 2$ and $t = 4$, so we could say that the area under the velocity curve from $t = a$ to $t = b$ is just $s(b) - s(a)$. The same logic applies.

Now let's write this using antiderivative notation. We know that the displacement is the area under the velocity curve, but it is *also* an antiderivative of the velocity. So we write

$$\int_a^b v(t)\,dt = s(b) - s(a),$$

which we read as "the area under the velocity curve from time $t = a$ to $t = b$ is equal to $s(b) - s(a)$." This is what we just observed, but we are using our new notation to describe it.

Thus we have the following geometric interpretation of derivatives and antiderivatives.

The...	is used for...
derivative	finding slopes of tangent lines.
antiderivative	finding areas under curves.

We observed this at the very beginning, but now we have developed calculus tools to find slopes and areas. It was easy to do this using formulas from geometry when we only considered constant or linear velocities, but now we can find slopes and areas for a very large group of functions – functions where there are no simple geometrical formulas to aid us.

By considering velocity and displacement, we were able to write

$$\int_a^b v(t)\,dt = s(b) - s(a).$$

But a similar statement can be made for other functions. The important point is that $s(t)$ is an antiderivative of $v(t)$. That's all we needed to make this work. Let's restate this in terms of x, since that's how it's usually stated.

Fundamental Theorem of Calculus, Part I

Let $f(x)$ be given, and suppose that $F(x)$ is an antiderivative of $f(x)$. Then for a and b in the domain of $f(x)$,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

A word on notation. An antiderivative written in the form

$$\int f(x)\,dx$$

is called an **indefinite integral,** and an antiderivative written in the form

$$\int_a^b f(x)\,dx$$

is called a **definite integral.** It is "definite" since you are specifying the interval $[a, b]$, so when no interval is specified, it is "indefinite."

Very often, the word "integrate" is used to mean the same thing as "antidiffereniate," and "integral" is used for "antiderivative."

Example 1

Find the area underneath $f(x) = \sin(x)$ and above the x-axis on the interval $[0, \pi]$, as shown in Figure 9. A good first step is to make a reasonable guess at the area. Looking at the right

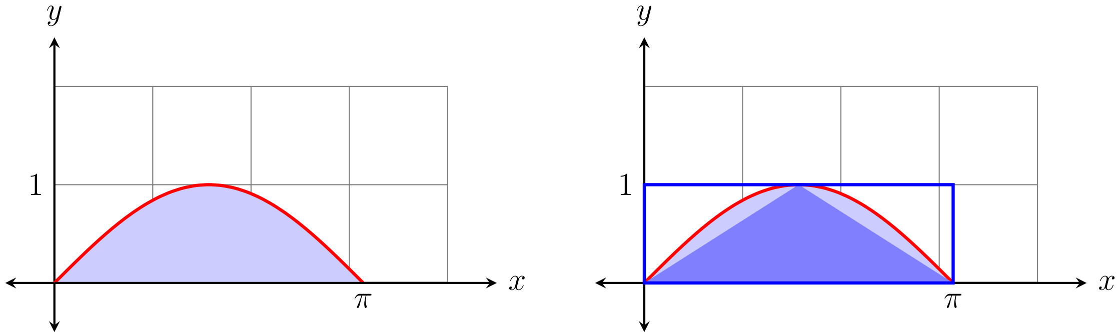


Figure 3: Area under $f(x) = \sin(x)$.

in Figure 9, you can see a dark blue triangle which fits inside the area, which has area

$$\frac{1}{2}(\pi)(1) \approx 1.57.$$

Further, the area sits inside the blue rectangle, which has area $\pi \approx 3.14$. So the total area under $\sin(x)$ is between 1.57 and 3.14, but closer to 1.57.

Now let's use calculus. We use The Fundamental Theorem of Calculus with $f(x) = \sin(x)$, $a = 0$, and $b = \pi$. Then

$$\int_0^\pi \sin(x)\,dx = F(\pi) - F(0),$$

where $F(x)$ is any antiderivative of $\sin(x)$. But we know that $-\cos(x)$ is an antiderivative of $\sin(x)$, and so

$$\begin{aligned}\int_0^\pi \sin(x)\,dx &= -\cos(\pi) - (-\cos(0)) \\ &= -(-1) - (-1) \\ &= 2.\end{aligned}$$

This fits well with our guesstimate, so we can be confident in our answer. The answer turned out to be very simple, but it is important to note that there is no simple geometrical formula we could have used to get 2. Calculus is really needed here, as it is for calculating most areas.

Example 2

Find the area of the shaded regions bounded by $f(x) = \sin(x)$ on the interval $[0, 2\pi]$, as shown in Figure 4.

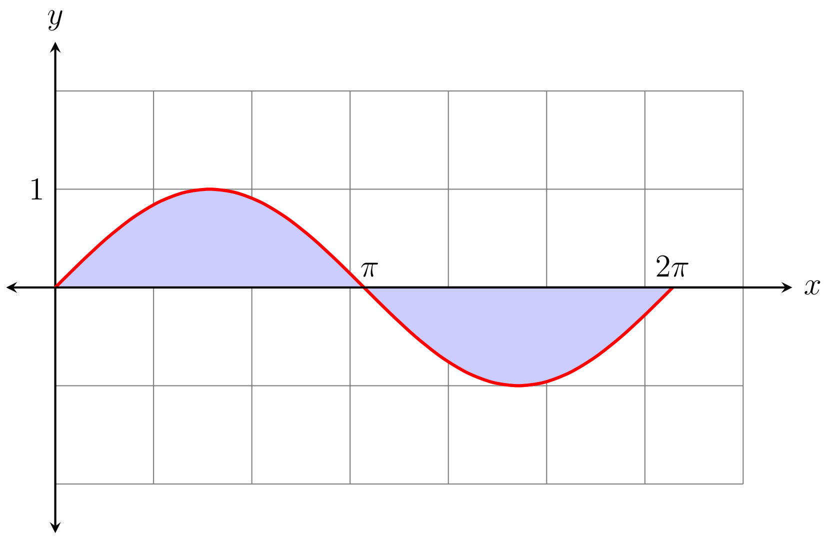


Figure 4: Area bounded by $f(x) = \sin(x)$.

We know from the previous problem that the answer should be 4, since there are two regions of area 2. But remember that regions *below* the x-axis make negative contributions to the area. So if we use the Fundamental Theorem of Calculus with $a = 0$ and $b = 2\pi$, we get

$$\begin{aligned}\int_0^{2\pi} \sin(x)\,dx &= -\cos(2\pi) - (-\cos(0)) \\ &= -1 - (-1) \\ &= 0.\end{aligned}$$

In other words, the areas cancel each other out, since there are two identically shaped regions, but one lies above the x-axis, and the other lies below the x-axis.

Example 3

Find the area below the curve $f(x) = 4 - x^2$ and above the x-axis, as shown in Figure 5.

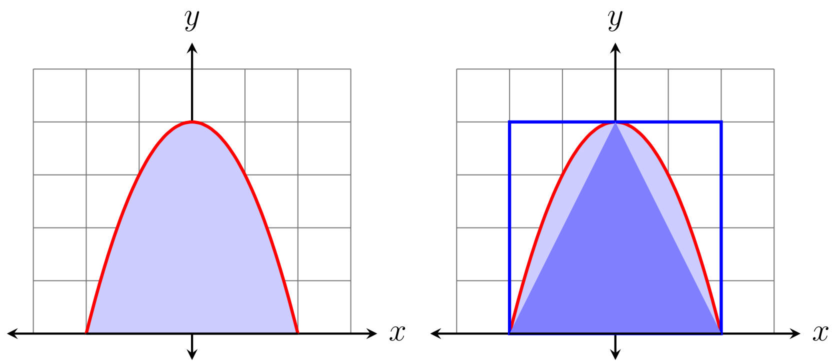


Figure 5: Area bounded by $f(x) = 4 - x^2$.

We can use the Fundamental Theorem of Calclulus here with $f(x) = 4 - x^2$. From the figure, it looks like we can use $a = -2$ and $b = 2$. To confirm this, we need to see where $f(x)$ crosses the x-axis. Solving $f(x) = 4 - x^2 = 0$ gives us $x = \pm 2$.

We can make a few guesstimates by looking at the right of Figure 5. The area is larger than the area of the blue triangle, which is $(4 \cdot 4)/2 = 8$, but smaller than the area of the blue rectangle, which is 16. We can visually see that it should be closer to 8 than to 16. To use the Fundamental Theorem of Calculus, we need an antiderivative of $f(x) = 4 - x^2$. Using the Inverse Power Rule, we can use $F(x) = 4x - \dfrac{1}{3}x^3$. Then

$$\begin{aligned}
\int_{-2}^{2} (4 - x^2)\, dx &= F(2) - F(-2) \\
&= 4 \cdot 2 - \frac{1}{3} \cdot 2^3 - \left(4(-2) - \frac{1}{3}(-2)^3\right) \\
&= 8 - \frac{8}{3} - \left(-8 - \left(\frac{-8}{3}\right)\right) \\
&= 8 - \frac{8}{3} + 8 - \frac{8}{3} \\
&= 16 - \frac{16}{3} \\
&= \frac{48}{3} \\
&= \frac{32}{3} \\
&\approx 10.7
\end{aligned}$$

This is consistent with our guesstimates. Here, finding an antiderivative is the easy part. It takes several steps to evaluate $F(2) - F(-2)$, so you need to be very careful.

There is an alternative way to approach this problem which cuts down on the algebra. As you can see on the right of Figure 6, the region we're looking at is symmetrical, so we can find the area of *half* the region and then multiply by 2.

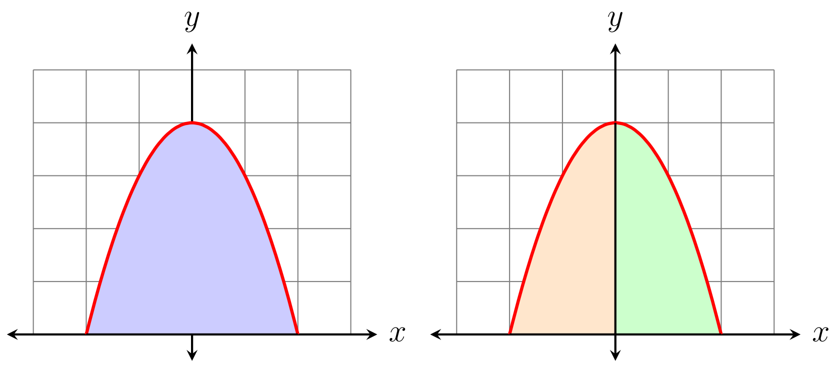


Figure 6: Using symmetry.

How will this help? The calculation is identical to the one we just made, except we use $a = 0$.

$$\begin{aligned}
\int_0^2 (4 - x^2)\, dx &= F(2) - F(0) \\
&= 4 \cdot 2 - \frac{1}{3} \cdot 2^3 - \left(4 \cdot 0 - \frac{1}{3} \cdot 0^3\right) \\
&= 8 - \frac{8}{3} - 0 \\
&= \frac{24}{3} - \frac{8}{3} \\
&= \frac{16}{3}
\end{aligned}$$

You can see how using $a = 0$ makes the calculations a lot simpler. Now we just need to double the area of one half of the region, and so the total area is $2 \cdot \frac{16}{3} = \frac{32}{3}$. As you can see, it is a good idea to take advantage of a symmetrical region so the calculations become easier.

Example 4

Find the area above the x-axis and below the curve $y = |x|$ on the interval $[-2, 3]$. See Figure 7.

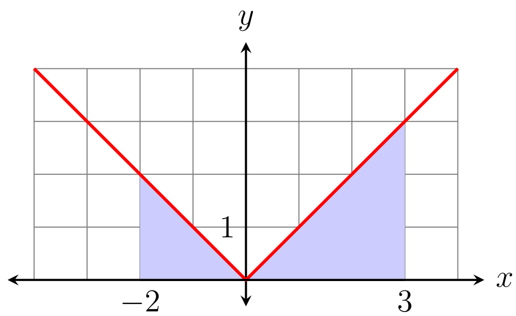


Figure 7: Area under $y = |x|$.

While we can just find the area of the triangles, let's see how using calculus would work. Thus, we want to find $\displaystyle\int_{-2}^{3} |x|\,dx$. The difficulty lies in finding an antiderivative of $|x|$. Since an antiderivative of x is $\frac{1}{2}x^2$, you might be tempted to choose $\left|\frac{1}{2}x^2\right|$. But since x^2 is always positive, then

$$\left|\frac{1}{2}x^2\right| = \frac{1}{2}x^2,$$

and so the derivative of $\left|\frac{1}{2}x^2\right|$ is x, *not* $|x|$.

So here, we have to go back to the piecewise definition of $y = |x|$:

$$y = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

This means we can break apart the integral into two separate parts, using the piecewise definition. Because we *do* know how to take antiderivatives of x and $-x$. And so

$$\int_{-2}^{3} |x|\,dx = \int_{-2}^{0} (-x)\,dx + \int_{0}^{3} x\,dx.$$

For the first integral, we use $F(x) = -\frac{1}{2}x^2$, so

$$\begin{aligned}\int_{-2}^{0} (-x)\,dx &= F(0) - F(-2) \\ &= -\frac{1}{2} \cdot 0^2 - \left(-\frac{1}{2}(-2)^2\right) \\ &= 0 - (-2) \\ &= 2.\end{aligned}$$

For the second integral, we use $F(x) = \dfrac{1}{2}x^2$, so

$$\begin{aligned}\int_0^3 x\,dx &= F(3) - F(0)\\ &= \frac{1}{2}\cdot 3^2 - \left(-\frac{1}{2}\cdot 0^2\right)\\ &= \frac{9}{2}.\end{aligned}$$

Putting it all together, we get

$$\begin{aligned}\int_{-2}^3 |x|\,dx &= \int_{-2}^0 (-x)\,dx + \int_0^3 x\,dx\\ &= 2 + \frac{9}{2}\\ &= \frac{4}{2} + \frac{9}{2}\\ &= \frac{13}{2}.\end{aligned}$$

What this example illustrates is that you can break up integrals if you have to. In other words, you can always use an intermediate point. Using integral notation, when $a \le b \le c$, then it is always the case that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Homework

1. Consider the region below the curve $f(x) = \sin(x) + \cos(x) + 2$ and above the x-axis on the interval $[0, 2\pi]$.

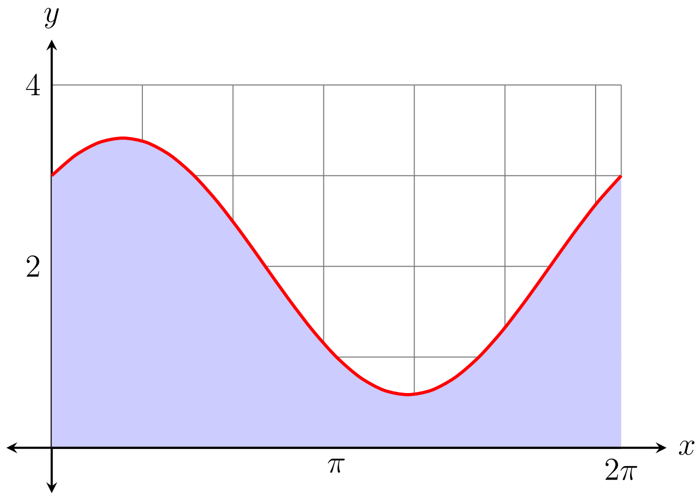


Figure 8: Area under $f(x) = \sin(x) + \cos(x) + 2$.

   (a) Visually, it looks like the area of this region is about half the area of the rectangular grid. Calculate this guesstimate for the area.

   (b) Using calculus, find the exact area. Is it close to your guesstimate?

2. Consider the function $f(x) = 16 - x^4$. Graph this on desmos. We will be looking at the area above the x-axis.

   (a) By drawing a triangle inside and a rectangle outside this region, find lower and upper guesstimates for the area. See Example 3.

   (b) By using symmetry appropriately, calculate the area of this region.

   (c) Verify that the area lies between your two guesstimates.

3. Consider the function defined below.

$$f(x) = \begin{cases} x+2, & x<1, \\ -x+4, & x \geq 1 \end{cases}$$

(a) Using basic geometry formulas, find the area bounded by $f(x)$ and the x-axis on the interval $[-4, 3]$. Be careful about negative areas.

(b) By writing the area as two separate integrals, compute the area using calculus.

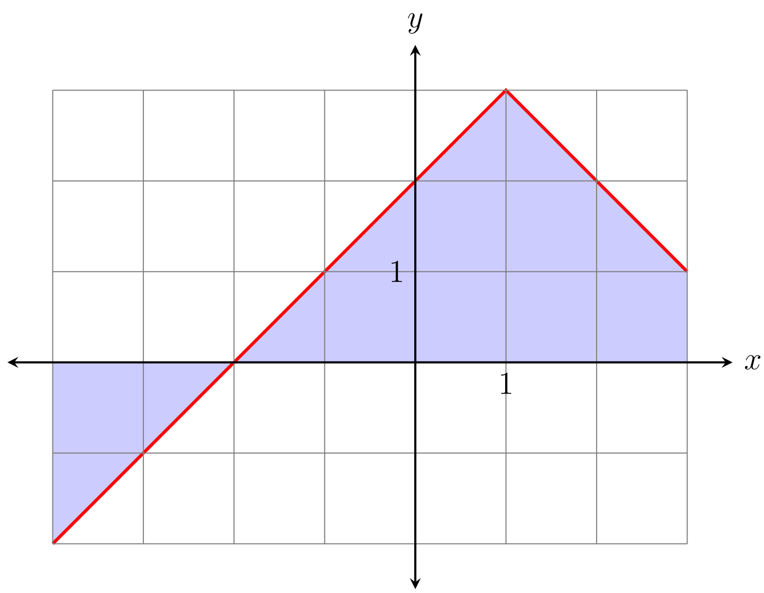


Figure 9: Area bounded by $f(x)$ and the x-axis.

Solutions

1. (a) The rectangle has dimensions 4 by 2π, and so half of this is

$$\frac{1}{2} \cdot 4 \cdot 2\pi = 4\pi.$$

(b) An antiderivative for $f(x)$ is $-\cos(x) + \sin(x) + 2x$, so the area is

$$\begin{aligned}\int_0^{2\pi} (\sin(x) + \cos(x) + 2)\, dx &= (-\cos(2\pi) + \sin(2\pi) + 2(2\pi)) - (-\cos(0) + \sin(0) + 2(0)) \\ &= (-1 + 0 + 4\pi) - (-1 + 0 + 0) \\ &= 4\pi.\end{aligned}$$

2. (a) The graph crosses the x-axis at -2 and 2, which you get by solving $16 - x^4 = 0$. So the triangle has a base of 4 and a height of 16, so its area is

$$\frac{1}{2} \cdot 4 \cdot 16 = 32.$$

The rectangle has a base of 4 and a height of 16, and so has area $4 \cdot 16 = 64$. Thus, the area of the region lies between 32 and 64.

(b) Since the region is symmetrical about the y-axis, we can find the area of half of the region and multiply by 2. An antiderivative for $16 - x^4$ is $16x - \frac{1}{5}x^5$.

$$\begin{aligned}2\int_0^2 (16 - x^4)\, dx &= 2\left(16(2) - \frac{1}{5} \cdot 2^5 - \left(16(0) - \frac{1}{5} \cdot 0^5\right)\right) \\ &= 2\left(32 - \frac{32}{5}\right) \\ &= 2\left(\frac{160}{5} - \frac{32}{5}\right) \\ &= \frac{256}{5} \\ &= 51.2.\end{aligned}$$

(c) We observe that $32 < 51.2 < 64$, so the area lies in the appropriate range.

3. (a) We can just count squares, or divide the region into triangles and trapezoids. We have an area of 2 below the x-axis, and an area of $\frac{17}{2}$ above the y-axis. Subtracting, we have an area of

$$\frac{17}{2} - 2 = \frac{17}{2} - \frac{4}{2} = \frac{13}{2}.$$

(b) We write the area as the sum of two integrals:

$$\int_{-4}^{1} (x+2)\,dx + \int_{1}^{3} (4-x)\,dx.$$

For the first, we have an antiderivative of $\frac{1}{2}x^2 + 2x$, so that

$$\begin{aligned}\int_{-4}^{1} (x+2)\,dx &= \left(\frac{1}{2}\cdot 1^2 + 2\cdot 1 - \left(\frac{1}{2}(-4)^2 + 2(-4)\right)\right) \\ &= \frac{1}{2} + 2 - 8 + 8 \\ &= \frac{5}{2}.\end{aligned}$$

For the second integral, we have an antiderivative of $4x - \frac{1}{2}x^2$, so that

$$\begin{aligned}\int_{1}^{3} (4-x)\,dx &= \left(4\cdot 3 - \frac{1}{2}\cdot 3^2 - \left(4\cdot 1 - \frac{1}{2}\cdot 1^2\right)\right) \\ &= 12 - \frac{9}{2} - 4 + \frac{1}{2} \\ &= 4.\end{aligned}$$

Adding these two areas, we get

$$\frac{5}{2} + \frac{8}{2} = \frac{13}{2}.$$