Antiderivatives, II

We just learned how we can use the process of antidifferentiation to solve everyday problems in physics. So far, we've found the displacement from the velocity when velocity is a linear function. It's time to go a bit further.

First, some common notation. We said the most general antiderivative of -9.8 was $-9.8t+C$, where the variable was t (for time), and C was whatever constant we chose. We write this as

$$\int -9.8\,dt = -9.8t + C.$$

The "$\int$" is the notation for taking an antiderivative, and the "dt" means the variable is t. If we were working with the variable were x, we would write

$$\int -9.8\,dx = -9.8x + C.$$

And instead of always saying "the most general antiderivative of $f(x)$," we just write

$$\int f(x)\,dx.$$

Learning to think backwards about differentiation does take a lot of practice.

Examples

We'll work out several short examples. It might be a good idea to have page 8 of the handout for Day 33 handy.

1. Find $\displaystyle\int x^3\,dx$.

   We know that when using the Power Rule to differentiate, we *decrease* the exponent by 1. So to antidifferentiate, we need to *increase* the exponent by one. But we don't just get x^4, because $\dfrac{d}{dx}x^4 = 4x^3$ – we have an extra factor of 4. So we compensate by dividing by 4, giving

   $$\int x^3\,dx = \frac{1}{4}x^4 + C.$$

   We can check by differentiating:

   $$\frac{d}{dx}\left(\frac{1}{4}x^4 + C\right) = \frac{1}{4}(4x^3) + 0 = x^3.$$

10 2. Find $\int x^n\,dx$.

We take the same approach as in the previous problem. Increasing the exponent by 1 gives x^{n+1}, but

$$\frac{d}{dx}x^{n+1} = (n+1)x^n.$$

So we get an extra factor of $n+1$, which we compensate for by dividing. Thus,

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

Again, we can check by differentiating:

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{1}{n+1}(n+1)x^n + 0 = x^n.$$

11 This rule is so important, we box it.

12 **Inverse Power Rule**

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1.$$

It is important to note why we must have $n \neq -1$. This is because we would get $\frac{1}{0}x^0$, which is undefined. But when $n = -1$, we have

$$\begin{aligned}\int x^{-1}\,dx &= \int \frac{1}{x}\,dx \\ &= \ln x + C,\end{aligned}$$

13 since we know that $\frac{d}{dx}\ln x = \frac{1}{x}$.

14 3. Find $\int (x^4 - 2x^3 + 5)\,dx$.

We apply the Inverse Power Rule.

$$\begin{aligned}\int (x^4 - 2x^3 + 5)\,dx &= \frac{1}{5}x^5 - 2\left(\frac{1}{4}x^4\right) + 5x + C \\ &= \frac{1}{5}x^5 - \frac{1}{2}x^4 + 5x + C.\end{aligned}$$

4. Find $\int \sqrt{x}\,dx$.

   As we did with derivatives, we rewrite as a power and then use the Inverse Power Rule.

$$\begin{aligned}\int \sqrt{x}\,dx &= \int x^{1/2}\,dx \\ &= \frac{1}{3/2}x^{3/2} + C \\ &= \frac{2}{3}x^{3/2} + C.\end{aligned}$$

5. Find $\int \frac{1}{x^6}\,dx$.

   Again, we must rewrite, as we needed to do with derivatives.

$$\begin{aligned}\int \frac{1}{x^6}\,dx &= \int x^{-6}\,dx \\ &= \frac{1}{-5}x^{-5} + C \\ &= -\frac{1}{5x^5} + C.\end{aligned}$$

   Be careful when *adding* 1 to negative exponents.

6. Find $\int \cos(x)\,dx$.

   Since $\frac{d}{dx}\sin(x) = \cos(x)$, then

$$\int \cos(x)\,dx = \sin(x) + C.$$

7. Find $\int \sin(x)\,dx$.

   The answer isn't $\cos(x) + C$, since $\frac{d}{dx}\cos(x) = -\sin(x)$. We have to compensate by putting in a negative sign.

$$\int \sin(x)\,dx = -\cos(x) + C.$$

8. Find $\int \frac{3}{x^2+1}\,dx$.

   We recognize $\frac{1}{x^2+1}$ as the derivative of $\arctan(x)$, so that

$$\int \frac{3}{x^2+1}\,dx = 3\arctan(x) + C.$$

22 9. Find $\displaystyle\int \frac{2}{\sqrt{1-x^2}}\,dx$.

We recognize the derivative of $\arcsin(x)$ here.

$$\int \frac{2}{\sqrt{1-x^2}}\,dx = 2\arcsin(x) + C.$$

Initial Value Problems

We've already seen initial value problems, such as in Example 2 in the handout for Day 34. In this example, we threw a marble down at 10 m/s. We used the fact that $a(t) = -9.8$, worked backwards (that is, took the antiderivative), and used the 10 to find $C = v_0$. Now the derivative of velocity is acceleration, and so $v'(t) = a(t)$. Let's redo this problem using our new notation.

We can rewrite this problem as follows.

Solve the initial value problem $v'(t) = -9.8,\ v(0) = 10$.

Here is the solution using new notation. There are no new steps or concepts involved here, just a different way of stating the problem (as an initial value problem) and writing the solution (using antiderivative notation).

$$\begin{aligned}
v(t) &= \int v'(t)\,dt \\
&= \int -9.8\,dt \\
&= -9.8t + C. \\
v(0) &= -9.8(0) + C \\
&= 10. \\
C &= 10. \\
v(t) &= -9.8t + 10.
\end{aligned}$$

Essentially, an initial value problem presents you with a derivative, but also some value of the function you're looking for. This additional information will allow you to find the $+C$.

Now let's look at some examples.

10. Solve the initial value problem $f'(x) = x^2 + 2x + 1,\ f(3) = 15$. We first take an antiderivative.

$$\begin{aligned}
f(x) &= \int (x^2 + 2x + 1)\,dx \\
&= \frac{1}{3}x^3 + 2\left(\frac{1}{2}x^2\right) + x + C \\
&= \frac{1}{3}x^3 + x^2 + x + C.
\end{aligned}$$

Then

$$\begin{aligned}
f(3) &= \frac{1}{3}\cdot 3^3 + 3^2 + 3 + C \\
&= 9 + 9 + 3 + C \\
&= 21 + C = 15,
\end{aligned}$$

Thus, $C = -6$, and so $f(x) = \dfrac{1}{3}x^3 + x^2 + x - 6$.

33 11. Solve the initial value problem $f'(x) = 3^x$, $f(0) = 1$.

Here, we need to remember that $\frac{d}{dx}3^x = 3^x \ln 3$, so to get a derivative of just 3^x, we need to divide by $\ln 3$.

$$f(x) = \int 3^x \, dx = \frac{3^x}{\ln 3} + C.$$

Let's confirm that dividing by $\ln 3$ was the right move.

$$\begin{aligned} \frac{d}{dx}\left(\frac{3^x}{\ln 3} + C\right) &= \frac{1}{\ln 3}(3^x \ln 3) + 0 \\ &= 3^x. \end{aligned}$$

Now we use the given fact $f(0) = 1$ to find C.

$$\begin{aligned} f(0) &= 1 \\ \frac{3^0}{\ln 3} + C &= 1 \\ \frac{1}{\ln 3} + C &= 1 \\ C &= 1 - \frac{1}{\ln 3} \end{aligned}$$

34 Thus, $f(x) = \frac{3^x}{\ln 3} + 1 - \frac{1}{\ln 3}$.

35 12. Solve the initial value problem $f'(x) = \sec^2(x) + \sin(x)$, $f(\pi/4) = 1$.

We need to remember that $\frac{d}{dx}\tan(x) = \sec^2(x)$. Then

$$\int (\sec^2(x) + \sin(x))\, dx = \tan(x) - \cos(x) + C.$$

Therefore,

$$\begin{aligned} f(\pi/4) &= 1 \\ \tan(\pi/4) - \cos(\pi/4) + C &= 1 \\ 1 - \frac{1}{\sqrt{2}} + C &= 1 \\ C &= \frac{1}{\sqrt{2}} \end{aligned}$$

36 So we get $f(x) = \tan(x) - \cos(x) + \frac{1}{\sqrt{2}}$.

As we saw with projectile motion – a common example in physics – we knew the acceleration, and used antidifferentiation to find the displacement. Let's return to Example 2 in the handout for Day 33, where we throw down a marble off a 20 meter high roof at 10 m/s. Restated as an initial value problem, we have

$$\text{Solve the initial value problem } s''(t) = -9.8,\ s'(0) = -10,\ s(0) = 20.$$

Since $s(t)$ is the displacement, $s''(t)$ is the acceleration, which is constant. Since the velocity is $s'(t)$, the statement $s'(0) = -10$ means that we are throwing the marble down at 10 m/s. And $s(0) = 20$ means we are throwing it from a height of 20 m. So the entire problem is restated using $s(t)$ *only.* This is the way such problems are usually stated in physics.

13. Solve the initial value problem $s''(t) = t^4 - t^2$, $s'(0) = 5$, $s(0) = 10$.

    Since we are given a *second* derivative, we have to antidifferentiate twice – first to find $s'(t)$, and then to find $s(t)$.

    $$s'(t) = \int (t^4 - t^2)\, dt = \frac{1}{5}t^5 - \frac{1}{3}t^3 + C.$$

    We use the information $s'(0) = 5$ to find C.

    $$5 = s'(0) = \frac{1}{5} \cdot 0^5 - \frac{1}{3} \cdot 0^3 + C,$$

    so that $C = 5$ and $s'(t) = \frac{1}{5}t^5 - \frac{1}{3}t^3 + 5$. We now antidifferentiate again to find $s(t)$.

    $$\begin{aligned} s(t) &= \int \left(\frac{1}{5}t^5 - \frac{1}{3}t^3 + 5\right) dt \\ &= \frac{1}{5}\left(\frac{1}{6}x^6\right) - \frac{1}{3}\left(\frac{1}{4}t^4\right) + 5t + C \\ &= \frac{1}{30}x^6 - \frac{1}{12}t^4 + 5t + C. \end{aligned}$$

    It is clear that plugging 0 into $s(t)$ just gives back C, so $C = 10$ from the given information $s(0) = 10$. Thus, $s(t) = \frac{1}{30}t^6 - \frac{1}{12} + 5t + 10$.

No single step in any of these problems was especially tricky. What makes this section challenging is that you have to remember *all* of your derivative formulas. And because there are many small steps, you have to really pay attention to the algebra. There are a few mistakes commonly made, so I'll make a short list here.

1. Incorrectly rewriting as powers of x, as in $\sqrt{x} = x^{1/2}$ and $\dfrac{1}{x^3} = x^{-3}$,
2. Using the Inverse Power Rule with $\dfrac{1}{x} = x^{-1}$, since $n = -1$ is not allowed. Instead, notice that $\dfrac{1}{x}$ is the derivative of $\ln x$,
3. Using the wrong $+$ or $-$ when taking the antiderivatives of $\sin(x)$ and $\cos(x)$,
4. Making a calculation error when using initial values to find C.

Finding Antiderivatives

Here is a summary of all the antiderivatives we know (that is, you can just use them at any time without justification), and the basic rules of antidifferentiation.

1. $\displaystyle\int \cos(x)\,dx = \sin(x).$
2. $\displaystyle\int \sin(x)\,dx = -\cos(x) + C.$
3. $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C.$
4. $\displaystyle\int e^x\,dx = e^x + C.$
5. $\displaystyle\int \frac{1}{x}\,dx = \ln x + C.$
6. $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b} + C.$
7. $\displaystyle\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x) + C.$
8. $\displaystyle\int \frac{1}{x^2+1} = \arctan(x) + C.$
9. The Inverse Power Rule:
   When $n \neq -1$,
   $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$
10. The Sum Rule:
   $$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx + C.$$
11. The Difference Rule:
   $$\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx + C.$$
12. The Constant Multiple Rule:
   $$\int cf(x)\,dx = c\int f(x)\,dx + C.$$

The Inverse Chain Rule: To integrate $\displaystyle\int f'(g(x))g'(x)\,dx$:

1. Look for a $g(x)$ and $g'(x)$ pair in the integrand – $g'(x)$ can be off by a constant multiple;
2. If $g'(x)$ is off by a constant multiple, multiply and divide by this constant and factor out;
3. Substitute $u = g(x)$, and solve for $du = g'(x)\,dx$;
4. Rewrite the integral in terms of u – all x's should disappear;
5. Find the antiderivative with respect to u;
6. Substitute back to rewrite in terms of x only.

Homework

1. Find $\int (x^5 - 6x^3 + x^2 - 4)\, dx$.
2. Find $\int \frac{1}{\sqrt{x}}\, dx$.
3. Find $\int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$.
4. Find $\int (3\sin(x) - 5\cos(x))\, dx$.
5. Find $\int \frac{6}{\sqrt{1-x^2}}\, dx$.
6. Solve the initial value problem $f'(x) = x^3 - 3x^2 - 1,\ f(2) = 10$.
7. Solve the initial value problem $f'(x) = \cos(x) - \sec^2(x),\ f(\pi/3) = -\sqrt{3}$.
8. Solve the initial value problem $s''(t) = t^3 + t$, $s'(0) = 3$, $s(0) = 8$.