Antiderivatives

As we just saw in the YouTube video, the trajectory of water shooting out from a pump looks like an upside-down parabola. Why should this be?

The answer is "gravity." If there wasn't any gravity, then when water shot out of a pump, it would just keep traveling in a straight line, going higher and higher. This is called Newton's First Law of Motion. Or if you threw a baseball, it would never hit the ground, it would just keep going in the direction you threw it.

Basically, what's happening is this. When you throw something up, it wants to keep going up. But gravity wants to bring it back down. So you have two opposite forces – how hard you threw it, and how strong gravity is. It turns out that gravity wins.

Why is that? It's really hard to go to the moon – you have to counteract gravity. In order to leave the Earth's atmosphere, it turns out that you have to be going *at least* 25,000 miles an hour! The fastest baseball pitch has been clocked at 102 mph. Not even close.

We need a little more physics. Suppose a skydiver jumps out of an airplane. Once you jump, you start falling. Your *velocity* keeps increasing as you keep falling – you keep falling faster and faster. But your *accleration* is constant.

This is a remarkable fact, and physicists have been studying gravity for centuries. What's important for us is just the fact that this acceleration due to gravity is constant, at 9.8 m/s^2.

A word about units. If displacement is measured in meters and time is measured in seconds, then the velocity – the rate of change – is measured in m/s. This is the same as mph (miles per hour), except with a change of units. So since acceleration is the derivative (rate of change) of velocity, its units are m/s/s, or m/s^2.

So why is this important? We know that when we take the derivative of displacement, we get the velocity. And when we take the derivative of the velocity, we get acceleration. And we know that acceleration is constant. So how do we find displacement? We have to work *backwards.* Before, we knew displacement and used derivatives to get acceleration. Now, we know the acceleration and have to use **antiderivatives** to get the displacement.

What is an antiderivative? Since the derivative of $f(x) = x^2$ is $2x$, we say that x^2 is an antiderivative of $2x$. We say *an* antiderivative, because there is more than one. But if $g(x) = x^2 + 5$, then $g'(x)$ is *also* an antiderivative of $2x$. That's because when you take the derivative of a constant, you get 0. We usually say that *the* antiderivative of $2x$ is $x^2 + C$, where C can be any number.

Example 1

You are standing on the roof of a building which is 20 m tall (this is about 60 ft). You drop a marble from the roof. How long will it take to hit the ground?

One important note is that all objects, no matter how small or large, will take the same time to hit the ground – this is a fundamental principle in physics. Here, we ignore the effects of air resistance. If you dropped a feather, air resistance would slow it down. But if you dropped a marble and a bowling ball, they would take the same amount of time to reach the ground because air resistance would be negligible.

Let's create a coordinate system, as shown in Figure 1. As we did before, when working with displacement, we use $s(t)$ for displacement, $v(t)$ for velocity, and $a(t)$ for acceleration.

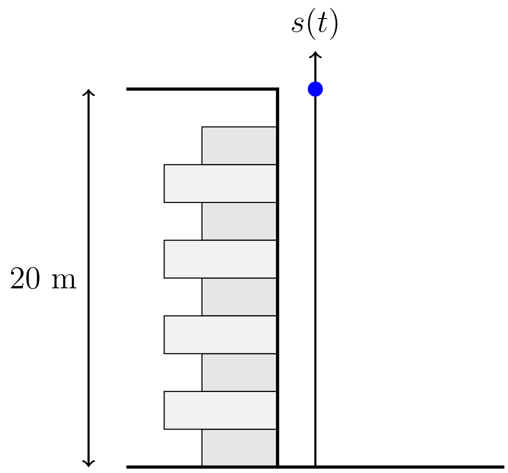

Figure 1: Dropping a marble from a roof.

Now let's start working backwards. We represent the fact that acceleration due to gravity is a constant 9.8 m/s^2 by

$$a(t) = -9.8.$$

Mathematically we have a negative acceleration because we are measuring displacement from the ground up. If we throw a ball *up*, gravity acts to bring it *down* – the opposite direction. The units of $a(t)$ are m/s^2.

Going backwards to find $v(t)$ – which is the antiderivative of velocity – we ask what function of t would we differentiate to get -9.8. Well, we know that the derivative of a linear function is constant, so we would guess

$$v(t) = -9.8t + C = -9.8t + v_0.$$

In physics, the term "v_0" is used instead of C, and is called the **initial velociy.** This is because

$$v(0) = -9.8(0) + v_0 = v_0.$$

The units of $v(t)$ are m/s.

What is the initial velocity in this problem? Since we are simply dropping the marble, it's just 0. Thus,

$$v(t) = -9.8t.$$

Time to go backwards again. We know that taking the derivative of t^2 will give us $2t$. So to just get t, we would have to start with $\frac{1}{2}t^2$. Now displacement is the antiderivative of velocity, and so

$$s(t) = -9.8\left(\frac{1}{2}t^2\right) + C = -4.9t^2 + s_0.$$

s_0 is called the **initial displacement** because

$$s(0) = -4.9(0^2) + s_0 = s_0.$$

In our case, we would use $s_0 = 20$, since the marble is being dropped from 20 m above ground. Thus,

$$s(t) = -4.9t^2 + 20.$$

The units of $s(t)$ are m.

Why was it so important to find $s(t)$? Our original question was to determine how long it took the marble to hit the ground. Since our coordinate system measures height above the ground, this is the same thing as asking when $s(t) = 0$, since 0 m above the ground is actually *on* the ground. So now we solve.

$$\begin{aligned} s(t) &= 0 \\ -4.9t^2 + 20 &= 0 \\ 4.9t^2 &= 20 \\ t^2 &\approx 4.08 \\ t &\approx 2.02 \text{ s} \end{aligned}$$

Note that we took the positive square root only as the time in seconds must be a positive number. So this means that the marble will hit the ground in approximately two seconds.

The important takeaway is that because of the *physics* of falling objects, we have to start with the acceleration and work *backwards* to find the displacement. In fact, much of calculus was created in order to explain physical phenomena. This is just one more example.

Example 2

You are standing on the roof of a building which is 20 m tall (this is about 60 ft). You throw a marble down from the roof at 10 m/s (about 22 mph), not unreasonable as it is five times slower that the fastest baseball pitch). (1) How long will it take to hit the ground? (2) At what velocity does it hit the ground?

We'll work through this one a bit more quickly as we have already seen the process. As before, we start with

$$a(t) = -9.8,$$

so that

$$v(t) = -9.8t + v_0.$$

Now what is v_0? We're throwing *down* at 10 m/s, and so v_0 is -10. Remember, we're measuring displacement as the distance *up* from the ground, so anything which acts to bring our marble *down* has to be negative. Therefore,

$$v(t) = -9.8t - 10.$$

Now we work backwards once more to find $s(t)$. We use $\frac{1}{2}t^2$ as an antiderivative of t, and $-10t$ as an antiderivative of -10. Thus,

$$\begin{aligned} s(t) &= -9.8\left(\frac{1}{2}t^2\right) - 10t + s_0 \\ &= -4.9t^2 - 10t + 20, \end{aligned}$$

where we again use 20 for s_0 since our building is 20 m tall.

So to answer the first question, we must find out when $s(t) = 0$, since that corresponds to being on the ground. But to solve

$$-4.9t^2 - 10t + 20 = 0,$$

we need to remember the quadratic formula. We'll state it with the variable t since that's what we're using.

Quadratic Formula

If $at^2 + bt + c = 0$, then

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To get accuracy to one decimal, we'll write numbers to two decimal places, then round to one decimal place for our final answer. We use $a = -4.9$, $b = -10$, and $c = 20$. Be very careful with negative signs.

$$
\begin{aligned}
t &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(-4.9)(20)}}{2(-4.9)} \\
&= \frac{10 \pm \sqrt{100 + 392}}{-9.8} \\
&= \frac{10 \pm 22.18}{-9.8} \\
\frac{10 + 22.18}{-9.8} &\approx -3.3 \\
\frac{10 - 22.18}{-9.8} &\approx 1.2
\end{aligned}
$$

As t represents a time, we choose the positive value. Thus, the marble hits the ground after about 1.2 s.

What is its velocity when it hits the ground? We substitute the value of t into the velocity equation, $v(t)$. Thus,

$$
\begin{aligned}
v(t) &= -9.8t - 10 \\
v(1.2) &= -9.8(1.2) - 10 \\
&\approx -21.8
\end{aligned}
$$

So the marble hits the ground at about -21.8 m/s, which is about -49 mph. To make sense of your answers, it is helpful to know that to convert m/s to mph, multiply by 2.237.

Now that we've worked out the displacement, let's do it one more time, just using v_0 and s_0 without substituting in values.

$$
\begin{aligned}
a(t) &= -9.8 \\
v(t) &= -9.8t + v_0 \\
s(t) &= -9.8\left(\frac{1}{2}t^2\right) + v_0 \cdot t + s_0 \\
&= -4.9t^2 + v_0 t + s_0.
\end{aligned}
$$

There's no need to work out all the steps each time. So we summarize.

Dispalcement Equations

If an object is thrown with an initial velocity of v_0 m/s from a height of s_0 m, the equations for the velocity and displacement are

$$v(t) = -9.8t + v_0, \quad s(t) = -4.9t^2 + v_0 t + s_0.$$

Just a few remarks. If you throw the object up, its initial velocity will be *postive.* We used a negative initial velocity because we were throwing it *down.* Also, it is helpful to know that if you are measuring velocity in ft/s, the acceleration due to gravity is -32 ft/s^2. Units for science are almost always metric, so we'll stick to m/s in our examples. You can easily convert back on forth with online unit converters. Just google "convert meters to feet" and you'll find one.

Example 3

Suppose you throw a baseball at an angle of $60°$ from the horizontal at a speed of 15 m/s (which is about 34 mph). When the baseball leaves your hand, it is 2 m above the ground. (1) When will it hit the ground again? (2) How far away will the ball land? (3) By finding an equation in x and y, show that the trajectory the baseball takes is a parabola.

Note: When working out **projectile motion** problems like this in physics, angles are usually measured in degrees, not radians.

In the previous examples, we were only concerned with vertical displacement. Now we're adding in horizontal displacement as well. Because we have two displacements, we'll call the vertical displacement $y(t)$ and the horizontal displacement $x(t)$, illustrated in Figure 2.

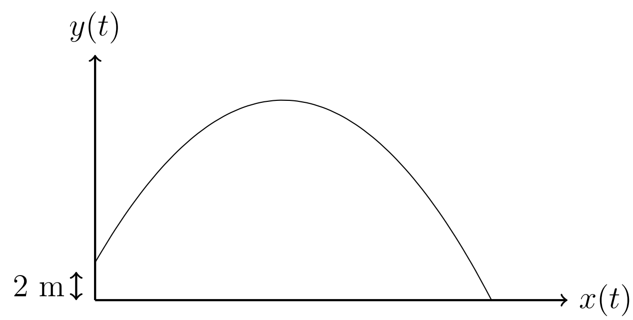

Figure 2: Throwing a baseball.

We know the formula for the vertical displacement from our previous work:

$$y(t) = -4.9t^2 + v_0 t + s_0.$$

What about the horizontal displacement? How does gravity affect the horizontal displacement? Not at all, actually. This is because gravity is a vertical force. A fundamental principle in physics is that a force in one direction has *no effect* on something moving in a *perpendicular* direction. Think about it: if you're running a race, does gravity slow you down? No, it doesn't. That's because your direction of motion is perpendicular to the force of gravity.

Since in our coordinate system, the x- and y-axes are perpendicular and gravity affects the y-direction, it has no effect on movement in the x-direction. In other words, the baseball moves with **constant horizontal velocity,** which we will call v_h. So when we figure out just what v_h is, then we can say that $x(t) = v_h t$. Summarizing, we have

$$y(t) = -4.9t^2 + v_0 t + s_0, \quad x(t) = v_h t.$$

We have three constants we have to figure out: v_0, s_0, and v_h. We know that $s_0 = 2$ as it is given in the problem. To figure out v_0 and v_h, we have to **decompose** the velocity into its vertical and horizontal components, as shown in Figure 3.

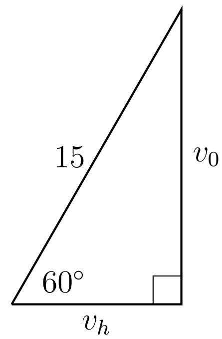

Figure 3: Decomposing a vector into vertical and horizontal components.

To do this, we create a right triangle as shown, and use trigonometry to find the lengths of the legs of the triangle. This decomposition of forces into components is another fundamental principle of physics. Reading off the triangle, we have

$$\sin 60^\circ = \frac{v_0}{15}, \quad \cos 60^\circ = \frac{v_h}{15}.$$

Then we get

$$v_0 = 15 \sin 60^\circ \approx 13, \quad v_h = 15 \cos 60^\circ = 7.5.$$

Thus, the vertical component of the velocity is about 13 m/s, and the horizontal component of the velocity is 7.5 m/s. So now we have equations for vertical and horizontal displacement:

$$y(t) = -4.9t^2 + 13t + 2, \quad x(t) = 7.5t.$$

We point out that we use +13 as we are throwing the ball *up.* In the previous example, we were throwing the marble *down.*

1. To see when the baseball will hit the ground, we solve $y(t) = 0$, since $y(t)$ is the vertical displacement:

   $$-4.9t^2 + 13t + 2 = 0.$$

   As before, we use the quadratic formula and choose the positive solution, which is $t \approx 2.8$ s. Note that we do not need to use $x(t)$ here because we are looking at vertical displacement.

2. To see how far away the baseball lands, we are looking for the *horizontal* displacement. Remember, gravity will bring the ball back down, but will have *no* effect on the horizontal displacement. So we plug 2.8 into $x(t)$, giving
$$x(2.8) = 7.5(2.8) = 21.$$
Thus, the baseball lands 21 m away from where you threw it.

3. To show that the baseball's trajectory is a parabola, we need to use the displacement equations,
$$y(t) = -4.9t^2 + 13t + 2, \quad x(t) = 7.5t,$$
and eliminate the variable t. This is not hard to do, since dividing the second equation by 7.5 gives $t = \dfrac{x}{7.5}$. Plugging back into the first equation:
$$\begin{aligned} y(t) &= -4.9t^2 + 13t + 2 \\ y &= -4.9\left(\frac{x}{7.5}\right)^2 + 13\left(\frac{x}{7.5}\right) + 2 \\ y &= -0.087x^2 + 1.73x + 2 \end{aligned}$$
Since the coefficient of x^2 is negative, this is the equation of a parabola which opens down.

Example 4

Suppose we want to create a circular fountain like in the video, where water shoots from spouts on the edge of the circle at ground level, and they all end up splashing in the center. We would like the spouts to shoot water at a 55° angle, and the diameter of the fountain is 50 m. (1) How fast does the water have to be shot out of the spouts for the waterspouts to converge in the center? (2) How high does the water go? Ignore any air resistance in this problem.

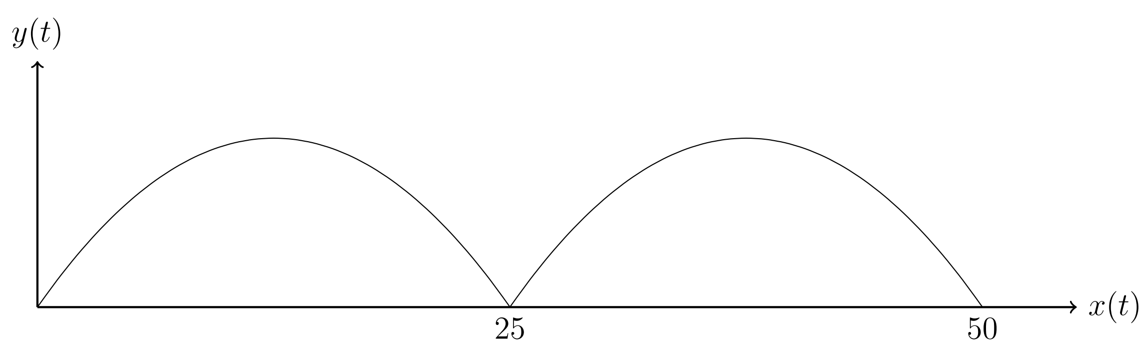

Figure 4: Planning converging fountains.

Let's rewrite the displacement equations for reference.
$$y(t) = -4.9t^2 + v_0t + s_0, \quad x(t) = v_ht.$$
Like before, we need to find v_0, s_0, and v_h. Since the spouts are on the ground, we know that $s_0 = 0$.

Now let's draw what is called a **force diagram** in physics, like we did before. Here, we don't know the velocity, so we represent it by v. Reading off the right triangle, we have

$$\sin 55^\circ = \frac{v_0}{v}, \quad \cos 55 = \frac{v_h}{v}.$$

Thus,

$$v_0 = v \sin 55^\circ \approx 0.82v, \quad v_h = v \cos 55^\circ \approx 0.57v.$$

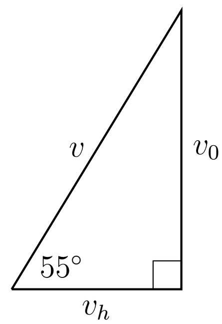

Figure 5: Decomposing a force into vertical and horizontal components.

Substituting back into the displacement equations (remember that $s_0 = 0$), we have

$$y(t) = -4.9t^2 + 0.82vt, \quad x(t) = 0.57vt.$$

Remember, v is a constant here.

Let's take stock of what we have. Here's the information we haven't used yet: the fact that the water has to travel 25 m in the horizontal direction to make it to the center, since the circle has a diameter of 50 m, and the fact that when the water *does* make it to the center, $y(t) = 0$.

First, we'll look at $y(t) = 0$. Note that because $s_0 = 0$, we won't need the quadratic formula.

$$\begin{aligned}
y(t) &= 0 \\
-4.9t^2 + 0.82vt &= 0 \\
t(0.82v - 4.9t) &= 0 \\
t &= 0 \\
0.82v - 4.9t &= 0 \\
0.82v &= 4.9t \\
t &= \frac{0.82v}{4.9} \\
&\approx 0.17v
\end{aligned}$$

Which value of t do we choose? The value $t = 0$ corresponds to the fact that the water shoots off from ground level, so we want $t = 0.17v$. This means once we figure out what v is, we can tell how long it takes the water to hit the center of the fountain.

So we use the last piece of information: the diameter of the fountain is 50 m. That means that at $t = 0.17v$ – when the water hits the center – it has traveled 25 m. We say this algebraically as

$$x(t) = x(0.17v) = 25.$$

Let's solve.

$$\begin{aligned} x(0.17v) &= 25 \\ (0.57v)(0.17v) &= 25 \\ 0.097v^2 &= 25 \\ v^2 &= \frac{25}{0.97} \\ v &= \sqrt{\frac{25}{0.97}} \\ &\approx 16.1 \end{aligned}$$

Therefore, we need the water to be shot out at 16.1 m/s (about 36 mph) so that it hits the center of the fountain exactly. Note that we choose the *positive* square root since the water is being shot *up* at an angle of 55°.

How high does the water go? Remember that when we know where the parabola opening down crosses the x-axis, the highest point occurs at the midpoint of those crossings. In other words, the water is at its highest point at $\frac{25}{2} = 12.5$ m.

But $y(t)$ is a function of t. So we *cannot* plug in 12.5 for t, since t is measured in *seconds,* not meters. So we have to go back to our equation for $x(t)$, solving $x(t) = 12.5$.

$$\begin{aligned} x(t) &= 0.57vt \\ &= (0.57)(16.1)t \\ &\approx 9.18t \\ 9.18t &= 12.5 \\ t &= \frac{12.5}{9.18} \\ &\approx 1.36 \end{aligned}$$

So the waterspouts reach their highest point at about 1.36 s. Plugging back into $y(t)$, we have

$$\begin{aligned} y(t) &= -4.9t^2 + 0.82vt \\ y(1.36) &= -4.9(1.36)^2 + 0.82(16.1)(1.36) \\ &\approx 8.9 \end{aligned}$$

So the highest the water goes is about 8.9 m, which is about 29 ft.

Homework

1. You are standing on the roof of a building which is 30 m tall. You accidentally drop your phone from the roof. How long will it take to hit the ground?

2. You are standing on the roof of a building which is 30 m tall. You throw a marble *up* from the roof at 12 m/s. (1) How long will it take to hit the ground? (2) At what velocity does it hit the ground?

3. Suppose you throw a baseball at an angle of 50° from the horizontal at a speed of 20 m/s. When the baseball leaves your hand, it is 2 m above the ground. (1) When will it hit the ground again? (2) How far away will the ball land? (3) By finding an equation in x and y, show that the trajectory the baseball takes is a parabola.

Solutions

1. We begin with the Displacement Equations, using $v_0 = 0$ and $s_0 = 30$:

$$s(t) = -4.9t^2 + 30.$$

When your phone hits the ground, $s(t) = 0$. Solving,

$$\begin{aligned} s(t) &= 0 \\ -4.9t^2 + 30 &= 0 \\ 4.9t^2 &= 30 \\ t^2 &\approx 6.12 \\ t &\approx 2.47 \end{aligned}$$

Thus, your phone hits the ground after about 2.47 s. Note we only considered the positive square root as we are looking for a time.

2. We begin with the Displacement Equations, using $v_0 = 12$ and $s_0 = 30$:

$$v(t) = -9.8t + 12, \quad s(t) = -4.9t^2 + 12t + 30.$$

(a) To find out how long the marble will take to hit the ground, we solve $s(t) = 0$ using the quadratic formula.

$$\begin{aligned} t &= \frac{-12 \pm \sqrt{12^2 - 4(-4.9)(30)}}{2(-4.9)} \\ &= \frac{-12 \pm \sqrt{144 + 588}}{-9.8} \\ &= \frac{-12 \pm 27.06}{-9.8} \\ \frac{-12 + 27.06}{-9.8} &\approx -1.5 \\ \frac{-12 - 27.06}{-9.8} &\approx 4.0 \end{aligned}$$

As t represents a time, we choose the positive value. Thus, the marble hits the ground after about 4.0 s.

(b) To find the velocity when the marble hits the ground, we substitute the value of t into the velocity equation, $v(t)$. Thus,

$$\begin{aligned} v(t) &= -9.8t + 12 \\ v(1.2) &= -9.8(4.0) + 12 \\ &\approx -27.2 \end{aligned}$$

So the marble hits the ground at about -27.2 m/s.

3. We know the formula for the vertical and horizontal displacement from our previous work:
$$y(t) = -4.9t^2 + v_0 t + s_0, \quad x(t) = v_h t.$$
We are given that $s_0 = 2$, but we will need to decompose the initial velocity vector in order to find v_0 and v_h.

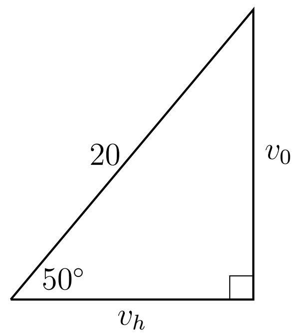

Figure 6: Decomposing a vector into vertical and horizontal components.

Reading off the triangle, we have
$$\sin 50^\circ = \frac{v_0}{20}, \quad \cos 50^\circ = \frac{v_h}{20}.$$
Then we get
$$v_0 = 20 \sin 50^\circ \approx 15.3, \quad v_h = 20 \cos 50^\circ \approx 12.9.$$
Thus, the vertical component of the velocity is about 15.3 m/s, and the horizontal component of the velocity is 12.9 m/s. So now we have equations for vertical and horizontal displacement:
$$y(t) = -4.9t^2 + 15.3t + 2, \quad x(t) = 12.9t.$$

(a) To see when the baseball will hit the ground, we solve $y(t) = 0$, since $y(t)$ is the vertical displacement:
$$-4.9t^2 + 15.3t + 2 = 0.$$
As before, we use the quadratic formula and choose the positive solution, which is $t \approx 3.2$ s.

(b) To see how far away the baseball lands, we are looking for the horizontal displacement. So we plug 3.2 into $x(t)$, giving
$$x(3.2) = 12.9(3.2) \approx 41.3.$$
Thus, the baseball lands 41.3 m away from where you threw it.

(c) To show that the baseball's trajectory is a parabola, we need to use the displacement equations,

$$y(t) = -4.9t^2 + 15.3t + 2, \quad x(t) = 12.9t,$$

and eliminate the variable t. Start by dividing the second equation by 12.9, giving $t = \dfrac{x}{12.9}$. Plugging back into the first equation:

$$\begin{aligned} y(t) &= -4.9t^2 + 15.3t + 2 \\ y &= -4.9\left(\frac{x}{12.9}\right)^2 + 15.3\left(\frac{x}{12.9}\right) + 2 \\ y &= -0.029x^2 + 1.19x + 2 \end{aligned}$$

Since the coefficient of x^2 is negative, this is the equation of a parabola which opens down.