Exam Hints, MA 131M, 16 November 2022

There will *definitely* be questions on implicit differentiation, deciding when to use L'Hôpital's Rule, and an arccos(cos(θ)) problem where θ is not in the range of arccos(x). There will be NO questions from the Summary of Limits in Calculus, as this involves too much material from the last two exams.

1. If $y^2 = \sin(xy)$, find $\dfrac{dy}{dx}$.

2. For each of the following, decide if it possible to use L'Hôpital's Rule. If you can, evaluate the limit by using it.

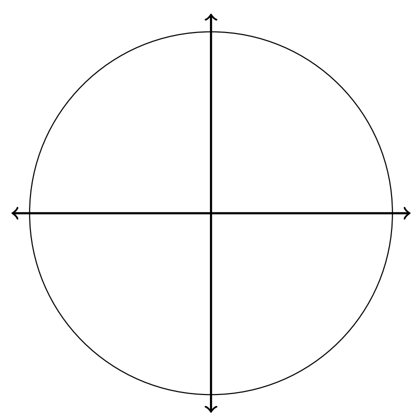
   (a) YES NO $\displaystyle\lim_{x\to 0} \frac{\cos^2(x) - 1}{x}$

   (b) YES NO $\displaystyle\lim_{x\to \infty} \frac{\sin(x)}{x^2}$

   (c) YES NO $\displaystyle\lim_{x\to -\infty} \frac{e^{-x}}{x}$

   (d) YES NO $\displaystyle\lim_{x\to \infty} \frac{\ln x}{x^{-1}}$

3. $\arccos(\cos(7\pi/6))$



4. Find the following limits.

   (a) $\displaystyle\lim_{x\to 0^+} \frac{x-2}{x\cos(x)}$

   (b) $\displaystyle\lim_{x\to 0^-} \frac{x-2}{x\cos(x)}$

   (c) $\displaystyle\lim_{x\to 0} \frac{x-2}{x\cos(x)}$

5. (a) Given that $x^3 + 3x + 2 = y^2$, find $\dfrac{dy}{dx}$.

   (b) Given that $x + y = x^3 + y^3$, find $\dfrac{dy}{dx}$.

   (c) Given that $x^2 y = \cos(y)$, find $\dfrac{dy}{dx}$.

6. Find $\dfrac{d}{dx} 6^{\cos(x)}$.

7. Find $\dfrac{d}{dx} \log_5(x^2 + 3x + 5)$.

8. Find $\lim\limits_{x \to \infty} xe^{-3x}$.

9. If $x^3 + xy = y$, find $\dfrac{dy}{dx}$.

10. Find $\sin(\arcsin(1/2))$.

11. Find $\arcsin(\sin(\pi))$.

12. Find $\dfrac{d}{dx} \arctan(2x)$.

13. Analyze the graph of $f(x) = \dfrac{x^2}{x+2}$ as done in class. You may look at the graph on desmos.

$$f'(x) = \frac{x^2 + 4x}{(x+2)^2}, \quad f''(x) = \frac{8}{(x+2)^3}.$$

Solutions

1.

$$
\begin{aligned}
y^2 &= \sin(xy) \\
\frac{d}{dx}y^2 &= \frac{d}{dx}\sin(xy) \\
2y\frac{dy}{dx} &= \cos(xy)\frac{d}{dx}xy \\
2y\frac{dy}{dx} &= \cos(xy)\left(x\frac{dy}{dx} + y\right) \\
2y\frac{dy}{dx} &= x\cos(xy)\frac{dy}{dx} + y\cos(xy) \\
2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} &= y\cos(xy) \\
(2y - x\cos(xy))\frac{dy}{dx} &= y\cos(xy) \\
\frac{dy}{dx} &= \frac{y\cos(xy)}{2y - x\cos(xy)}
\end{aligned}
$$

2. (a) As the limit is of the form $\frac{0}{0}$, L'Hôpital's Rule may be applied here. The Chain Rule is needed for the numerator.

$$\lim_{x\to 0}\frac{\cos^2(x) - 1}{x} \overset{\text{LR}}{=} \lim_{x\to 0}\frac{2\cos(x)(-\sin(x))}{1} = 0.$$

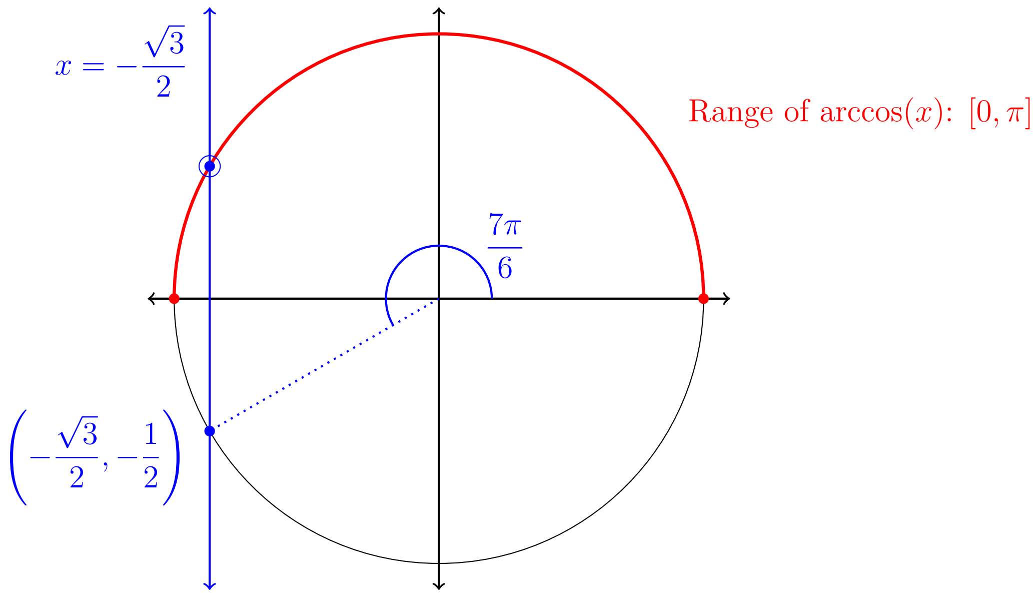
(b) L'Hôpital's Rule does not apply since $\lim_{x\to\infty}\sin(x)$ DNE. Since the denominator is of the form ∞, the numerator would also have to be of the form ∞ for L'Hôpital's Rule to apply. But $\sin(x)$ has no limit as $x \to \infty$, and never gets larger than one.

(c) This limit is of the form $\frac{\infty}{\infty}$, and so L'Hôpital's Rule applies.

$$\lim_{x\to -\infty}\frac{e^{-x}}{x} \overset{\text{LR}}{=} \lim_{x\to -\infty}\frac{-e^{-x}}{1} \text{ DNE } (-\infty).$$

(d) The limit is of the form $\frac{\infty}{0}$, and so L'Hôpital's Rule may not be applied.

3. By looking at the graph below, we see that $\arccos(\cos(7\pi/6)) = 5\pi/6$.



4. All limits are of the form $\dfrac{-2}{0}$ and so DNE.

   (a) As $x \to 0^+$, both $x > 0$ and $\cos(x) > 0$, so the fraction is of the form $\dfrac{-}{++}$. So the limit is DNE ($-\infty$).

   (b) As $x \to 0^-$, then $x < 0$ and $\cos(x) > 0$, so the fraction is of the form $\dfrac{-}{-+}$. So the limit is DNE ($+\infty$).

   (c) As the limits from the left and right are different, this limit is just DNE.

5. (a) This problem is worked out on page 79 of the ebook (see Day 1).

   (b) This problem is worked out on page 80 of the ebook (see Day 1).

   (c)

$$\begin{aligned}
x^2 y &= \cos(y) \\
\frac{d}{dx} x^2 y &= \frac{d}{dx} \cos(y) \\
x^2 \frac{dy}{dx} + 2xy &= -\sin(y) \frac{dx}{dy} \\
2xy &= -\sin(y) \frac{dy}{dx} - x^2 \frac{dy}{dx} \\
2xy &= -(\sin(y) + x^2) \frac{dy}{dx} \\
\frac{dy}{dx} &= -\frac{2xy}{\sin(y) + x^2}
\end{aligned}$$

6. $-\sin(x) 6^{\cos(x)}$.

7. $\dfrac{2x+3}{(x^2+3x+5)\ln 5}$.

8. 0.

9. $\dfrac{3x^2+y}{1-x}$.

10. $\dfrac{1}{2}$.

11. 0.

12. $\dfrac{2}{4x^2+1}$.

13. (a) No horizontal asymptotes; vertical asymptote at $x = -2$.
    (b) Local maximum at $(-4, -8)$, local minimum at $(0, 0)$.
    (c) Increasing on $(-\infty, -4)$ and $(0, \infty)$. Decreasing on $(-4, -2)$ and $(-2, 0)$.
    (d) There are no points of inflection.
    (e) Concave down on $(-\infty, -2)$ and concave up on $(-2, \infty)$.