Remember that the Final Exam will contain two question from each of the first three exams. Here are problems from the first two exams which correspond to the review we did in class today.

1. Find the equation of the line tangent to $y = x^3 + x^2$ at $x = 1$. Check your work by graphing on desmos.

   Solution: Let $f(x) = x^3 + x^2$. Then $f'(x) = 3x^2 + 2x$, and so the slope of the tangent line is $m = f'(1) = 5$. Since $f(1) = 2$, we use the point $(1, 2)$. Then

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 5(x - 1) \\ y - 2 &= 5x - 5 \\ y &= 5x - 3 \end{aligned}$$

2. The population of a colony of bacteria at time t (in hours) is given by the equation $P(t) = 3000e^{0.01t}$. Find out how fast the colony is growing at time $t = 3$.

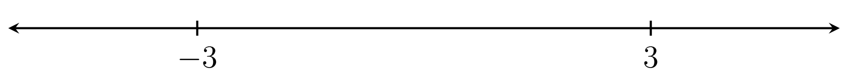
   Solution: Since we're being asked for a rate of change, we evaluate $P'(3)$.

$$\begin{aligned} P(t) &= 3000e^{0.01t} \\ P'(t) &= 3000 \cdot 0.01e^{0.01t} \\ &= 30e^{0.01t} \\ P'(3) &= 30e^{0.01 \cdot 3} \\ &\approx 30.9 \end{aligned}$$

   So the colony is growing at about 31 bacteria per hour at time $t = 3$.

3. Suppose you are given $f(x) = \frac{1}{3}x^3 - 9x$, so that $f'(x) = x^2 - 9$. Using a **sign chart,** determine where the function is increasing and decreasing. Write your answer in **interval notation.** Visually check by graphing on desmos.

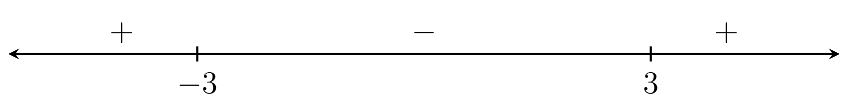
   Solution: First, we solve $f'(x) = x^2 - 9 = 0$, and so $x = -3, 3$. This divides the number line into three parts.



   Easy test point in each interval are -4, 0, and 4. We evaluate $f'(x)$ at these points.

$$f'(-4) = 7 > 0, \quad f'(0) = -9 < 0, \quad f'(4) = 7 > 0.$$

   We indicate this on the number line.



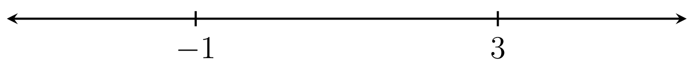
   Thus, $f(x)$ is increasing on $(-\infty, -3)$ and $(3, \infty)$, and decreasing on $(-3, 3)$.

4. Suppose you are given the following function and its derivatives:

$$f(x) = \frac{1}{4}x^4 - \frac{5}{3}x^3 + \frac{3}{2}x^2 + 9x$$
$$f'(x) = x^3 - 5x^2 + 3x + 9 = (x-3)^2(x+1)$$
$$f''(x) = 3x^2 - 10x + 3$$

Determine where $f'(x) = 0$. Using a **sign chart,** determine if there is a minimum, maximum, or inflection point at these values of x. Check with **desmos.**

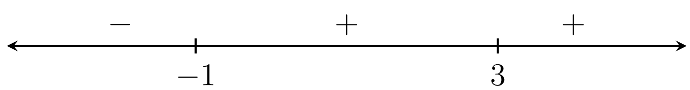
Solution: Solving $f'(x) = 0$ is not difficult since we are given it in factored form. This gives $x = -1$ and $x = 3$.



Easy test points are −2, 0, and 4.

$$f'(-2) = -25 < 0, \quad f'(0) = 9 > 0, \quad f'(4) = 5 > 0,$$

giving the following number line.



Since we go from decreasing to increasing at $x = -1$, there is a local minimum here. Since we are increasing on both side of $x = 3$, there is an inflection point here.

5. Find the global extrema of the function $f(x) = 4+4x-x^2$ on the closed interval $[-2, 4]$. Do *not* just make a graph and look at it. You *must* use the appropriate theorem from calculus. Visually check with desmos.

Solution: First, we find where $f'(x) = 0$ or is undefined. $f'(x) = 4 - 2x$, and so is defined everywhere. $f'(x) = 0$ occurs when $x = 2$.

Now evaluate $f(x)$ here and at the endpoints.

$$f(2) = 8, \quad f(-2) = -8, \quad f(4) = 4.$$

Thus, there is a global minimum at $(-2, -8)$ and a global maximum at $(2, 8)$.

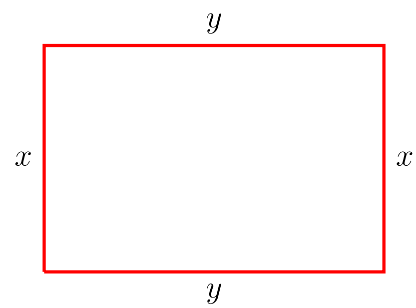
6. Show that the curves $y = x^2 + 2x$ and $y = 3 - x - x^2$ intersect on the interval $[0, 2]$. You *must* use the appropriate theorem from calculus, you *cannot* just look at the graph. Check with desmos.

   Solution: Put $f(x) = x^2 + 2x - (3 - x - x^2) = 2x^2 + 3x - 3$. Because $f(x)$ is a polynomial, it is continuous. Now evaluate

   $$f(0) = -3, \quad f(2) = 11.$$

   Since 0 is between -3 and 11, by the IVT, there is some x_0 in the interval $[0, 2]$ where $f(x_0) = 0$. The curves intersect at this point x_0.

7. Suppose you want to fence in a rectangular area, as shown, and have 300 m of fencing. What is the largest area you can enclose? Set up *only.* That is, give a function $f(x)$ and a closed interval $[a, b]$. Do *not* actually work out the largest area.



   Solution: Since we are looking for the largest area, we use $f(x) = xy$. Since the perimeter is $2x + 2y$, we can use this to solve for y.

   $$\begin{aligned} 2x + 2y &= 300 \\ 2y &= 300 - 2x \\ y &= 150 - x \end{aligned}$$

   Substituting in, we get

   $$f(x) = x(150 - x) = 150x - x^2.$$

   Since two sides of the fence are x, then x cannot be greater than 150. So an appropriate interval would be $[0, 150]$.

8. Evaluate the following limits, using L'Hôpital's Rule when appropriate.

   (a) $\lim_{x\to\infty} x^2 \ln(2x)$

   (b) $\lim_{x\to 0^+} x^2 \ln(2x)$

Solution: The first limit is of the form $+\infty \cdot +\infty$. Thus, this limit DNE ($+\infty$).

The second limit is of the form $0 \cdot -\infty$. We rewrite and then use L'Hôpital's Rule. Note that we may use a rule of logarithms to simplify $\ln(2x) = \ln(2) + \ln(x)$. Keep the logarithm on the numerator.

$$\begin{aligned}
\lim_{x\to 0^+} x^2(\ln(2)+\ln(x)) &= \lim_{x\to 0^+} \frac{\ln(2)+\ln(x)}{x^{-2}} \\
&\overset{\text{LR}}{=} \lim_{x\to 0^+} \frac{0+1/x}{-2x^{-3}} \\
&= \lim_{x\to 0^+} \frac{1/x}{-2/x^3} \\
&= \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{x^3}{-2} \\
&= \lim_{x\to 0^+} -\frac{x^2}{2} \\
&= 0
\end{aligned}$$

Working with the Definition of the Derivative

There is usually quite a bit of algebra in working with the definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

So let's break it down into steps.

Let's start with $f(x) = x^2 - 3x$. How do we evaluate $f(x+h)$? We substitute $x+h$ everywhere we see an x. If you sometimes get stuck with this, here's something to try.

1. First, rewrite the function with boxes.

$$f\left(\boxed{}\right) = \left(\boxed{}\right)^2 - 3\left(\boxed{}\right).$$

2. Next, put $x+h$ in each empty box.

$$f\left(\boxed{x+h}\right) = \left(\boxed{x+h}\right)^2 - 3\left(\boxed{x+h}\right).$$

3. We don't need the boxes any more.

$$f(x+h) = (x+h)^2 - 3(x+h).$$

4. Expand. Be careful when distributing the minus sign.

$$f(x+h) = x^2 + 2xh + h^2 - 3x - 3h.$$

5. Now substitute into the limit definition and simplify until the h cancels. Again, watch the minus signs. Note that for the h to cancel, *every* term in the numerator that does *not* contain h should cancel.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - (x^2 - 3x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} \\
&= \lim_{h\to 0} \frac{h(2x + h - 3)}{h} \\
&= \lim_{h\to 0} (2x + h - 3) \\
&= 2x - 3.
\end{aligned}$$

You will have a simple quadratic using the definition of the derivative on the Final Exam. It's easy to make up one on your own and try it. Make sure you use at least one negative sign, since you can certain I will, too.