Calculus and Graphing

Using calculus to study graphs has been a common theme throughout the course. We've looked at different aspects of graphs at different times. Here, we look at some examples and apply everything we've learned. We'll look at polynomials and rational functions. For each graph, we will:

1. Determine horizontal and vertical asymptotes, if any;
2. Determine local minima and maxima, if any;
3. Determine intervals where the function is increasing and decreasing;
4. Determine inflection points, if any;
5. Determine intervals on which the graph is concave up or concave down.

We will not focus on the algebra of derivatives of rational functions – this can get hairy. So we'll use software to calculate the derivatives for us. Also, we won't discuss how to determine the information if you don't have a graph. The idea is that you *do* have a graph – and any time you'd need a graph of a function in real life, you would use software. So the emphasis here is on describing various features of a graph using Calculus.

We will be working with desmos.com to aid in our explorations.

Example 1

Our first example is the polynomial $f(x) = 3x^5 - 5x^3$, which is ◯1. The first two derivatives are:

$$f'(x) = 15x^4 - 15x^2, \quad f''(x) = 60x^3 - 30x.$$

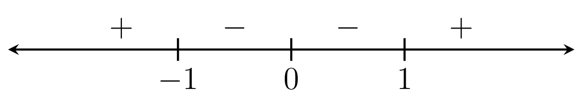
1. Asymptotes. We know that polynomials *never* have any asymptotes. Their behavior as $x \to -\infty$ and $x \to \infty$ is determined by the highest degree term.

2. Local Minima and Maxima. We need to solve $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 15x^4 - 15x^2 &= 0 \\ 15x^2(x^2 - 1) &= 0 \\ 15x^2(x+1)(x-1) &= 0 \\ x &= -1, 0, 1 \end{aligned}$$

Since $f''(-1) = -30 < 0$, the graph is concave down at $x = -1$, so there is a local maximum at $(-1, 2)$. Since $f''(1) = 30 > 0$, the graph is concave up at $x = 1$, so there is a local minimum at $(1, 2)$. But at $x = 0$, we have $f''(0) = 0$, so we have to make a sign chart. This really isn't more work, since we need a sign chart to find out where the function is increasing and decreasing. We take test points of -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2. Evaluating:

$$f'(-2) = 180, \quad f'\left(-\frac{1}{2}\right) = -\frac{45}{16}, \quad f'\left(\frac{1}{2}\right) = -\frac{45}{16}, \quad f'(2) = 180.$$



So, since we are decreasing on both sides of $x = 0$, there is an inflection point at $(0, 0)$. All this can be visually verified by looking at the graph.

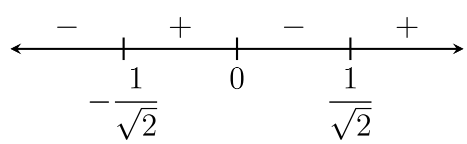
3. Intervals of Increase and Decrease. We've already created a sign chart for $f'(x)$, so we can read off the intervals where the graph is increasing and decreasing. The function is increasing on $(-\infty, -1)$ and $(1, \infty)$ since $f'(x) > 0$ there, and decreasing on $(-1, 1)$ since $f'(x) < 0$ there.

4. Inflection Points. To find possible inflection points, we solve $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ 60x^3 - 30x &= 0 \\ 30x(2x^2 - 1) &= 0 \\ x &= 0 \\ 2x^2 - 1 &= 0 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \\ x &= -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \end{aligned}$$

We know that there is an inflection point at $(0, 0)$ from previous work. Since $\frac{1}{\sqrt{2}} \approx 0.7$, we can make a sign chart with test points -1, $-\frac{1}{2}$, $\frac{1}{2}$, and 1. Evaluating:

$$f''(-1) = -30, \quad f''\left(-\frac{1}{2}\right) = \frac{15}{2}, \quad f''\left(\frac{1}{2}\right) = -\frac{15}{2}, \quad f''(1) = 30.$$



Since the concavity changes at $-1/\sqrt{2}$ and $1/\sqrt{2}$, then there are inflection points at the points $(-1/\sqrt{2}, 7/4\sqrt{2})$ and $(1/\sqrt{2}, -7/4\sqrt{2})$. Since we want to visually inspect the graph, you can use a calculator to approximate these points as $(-0.7, 1.2)$ and $(0.7, -1.2)$. Or select ◯2, which is the equation $x = \frac{1}{\sqrt{2}}$. If you zoom in on this point, you should be able to see that the concavity changes there.

5. Intervals of Concavity. Since we have a sign chart for $f''(x)$, we see that the graph is concave down on $(-\infty, -1/\sqrt{2})$ and $(0, 1/\sqrt{2})$ (since $f''(x) < 0$ there), and the graph is concave up on $(-1/\sqrt{2}, 0)$ and $(1/\sqrt{2}, \infty)$ (since $f''(x) > 0$ there).

Assessment Expectations: For a problem like this, you would be given a graph and both the derivatives. You would only be asked to do one or two out of the five parts of the problem. You will need to make a sign chart for at least one of the parts.

Example 2

We now tackle the rational function $f(x) = \dfrac{x}{x^2+1}$, which is ◯3. The first two derivatives are:

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}, \quad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

1. Asymptotes. The degree of the numerator is $N = 1$ and the degree of the denominator is $D = 2$. Since $N < D$, we know that $y = 0$ is a horizontal asymptote.

   Note that the denominator is *always* positive. Since it can never be 0, there are no vertical asymptotes.

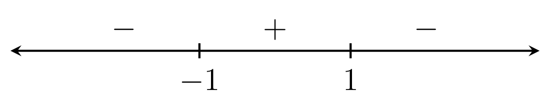
2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0.

$$\begin{aligned} 1 - x^2 &= 0 \\ (1+x)(1-x) &= 0 \\ x &= -1, 1 \end{aligned}$$

   Since $f''(-1) = 1/2 > 0$, the graph is concave up at $x = -1$, so there is a local minimum at $(-1, -1/2)$. Since $f''(1) = -1/2 < 0$, the graph is concave down at $x = 1$, so there is a local maximum there at $(1, 1/2)$. These points are easily visible on the graph.

3. Intervals of Increase and Decrease. Since we didn't need a sign chart for the local extrema, we make one now. Easy test points are -2, 0, and 2. Evaluating:

$$f'(-2) = -\frac{3}{25}, \quad f'(0) = 1, \quad f'(2) = -\frac{3}{25}.$$



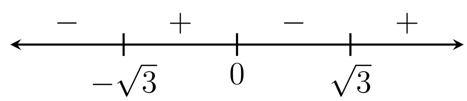
   Now we can read off the intervals where the graph is increasing and decreasing. The function is increasing on $(-1, 1)$ since $f'(x) > 0$ there, and decreasing on $(-\infty, -1)$ and $(1, \infty)$ since $f'(x) < 0$ there.

4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0.

$$\begin{aligned} 2x(x^2-3) &= 0 \\ x &= 0 \\ x^2-3 &= 0 \\ x^2 &= 3 \\ x &= -\sqrt{3}, \sqrt{3} \end{aligned}$$

Since $\sqrt{3} \approx 1.7$, we can choose test points -2, -1, 1, and 2. Evaluating:

$$f''(-2) = -\frac{4}{125}, \quad f''(-1) = \frac{1}{2}, \quad f''(1) = -\frac{1}{2}, \quad f''(2) = \frac{4}{125}.$$



Since the concavity changes at $-\sqrt{3}$, 0, and $\sqrt{3}$, then there are inflection points at the points $(-\sqrt{3}, -\sqrt{3}/4)$, $(0,0)$, and $(\sqrt{3}, \sqrt{3}/4)$. Since we want to visually inspect the graph, you can use a calculator to approximate two of these points as $(-1.7, -0.4)$ and $(1.7, 0.4)$. Or select ◯4, which is the equation $x = \sqrt{3}$. If you zoom in on this point, you should be able to see that the concavity changes there.

5. Intervals of Concavity. Since we already have a sign chart for $f''(x)$, we see that the graph is concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ (since $f''(x) < 0$ there), and the graph is concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ (since $f''(x) > 0$ there).

61 **Example 3**

We now explore the rational function $f(x) = \dfrac{x}{x^2-1}$, which is ⑤. The first two derivatives are:

$$f'(x) = -\frac{x^2+1}{(x^2-1)^2}, \quad f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}.$$

62 1. Asymptotes. The degree of the numerator is $N = 1$ and the degree of the denomi-
63 nator is $D = 2$. Since $N < D$, we know that $y = 0$ is a horizontal asymptote.

   For vertical asymptotes, we set the denominator equal to 0.

$$\begin{aligned} x^2 - 1 &= 0 \\ (x+1)(x-1) &= 0 \\ x &= -1, 1 \end{aligned}$$

   So there are vertical asymptotes at $x = -1$ and $x = 1$. Looking at the graph, we describe the behavior of the function at these asymptotes using limit notation:

$$\lim_{x\to -1^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x\to -1^+} f(x) \text{ DNE } (+\infty),$$

$$\lim_{x\to 1^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x\to 1^+} f(x) \text{ DNE } (+\infty).$$

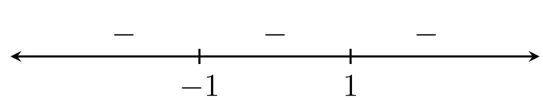
2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0.

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \end{aligned}$$

64 Since x^2 can never be negative, we see that there are no local extrema.

3. Intervals of Increase and Decrease. To make a sign chart, we need to use the points where $f'(x) = 0$ or where the function is undefined. Note that in the previous examples, we didn't have to consider where the function was undefined since both were defined for *all* real numbers. The only places where a rational function is undefined is where there are vertical asymptotes, in this case, -1 and 1. Easy test points are -2, 0, and 2. Evaluating:

$$f'(-2) = -\frac{5}{9}, \quad f'(0) = -1, \quad f'(2) = -\frac{5}{9}.$$



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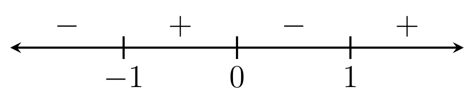
Since $f'(x)$ is negative everywhere it's defined, then the function is decreasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. It is important to note that you can't just say the function is decreasing on $(-\infty, \infty)$, since all real numbers includes -1 and 1, but the function is not defined at these points. So you have to write as three separate intervals.

4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0.

$$\begin{aligned} 2x(x^2+3) &= 0 \\ x &= 0 \\ x^2+3 &= 0 \\ x^2 &= -3 \qquad\qquad\qquad \text{impossible} \end{aligned}$$

So the only possible inflection point is at $x = 0$. We make a sign chart, but again we must include -1 and 1, since we have seen examples where the concavity changes as you hop over a vertical asymptote. Easy test points are -2, $-\frac{1}{2}$, $\frac{1}{2}$, and 2. Evaluating:

$$f''(-2) = -\frac{28}{27}, \quad f''\left(-\frac{1}{2}\right) = \frac{208}{27}, \quad f''\left(\frac{1}{2}\right) = -\frac{208}{27}, \quad f''(2) = \frac{28}{27}.$$



Since the concavity changes at 0, then there is an inflection points at the point $(0, 0)$. Concavity does change at $x = -1$ and $x = 1$, but the change is *over an asymptote.* Because $f(x)$ is not defined when $x = -1$ and $x = 1$, these cannot correspond to inflection points.

5. Intervals of Concavity. Since we already have a sign chart for $f''(x)$, we see that the graph is concave down on $(-\infty, -1)$ and $(0, 1)$ (since $f''(x) < 0$ there), and the graph is concave up on $(-1, 0)$ and $(1, \infty)$ (since $f''(x) > 0$ there).

Homework

Analyze the following functions as done above. The first one is fairly easy, so you can get a feel for working through the steps with minimal algebra. The second one is a little more involved. When working through these, make sure to include where the vertical asymptotes are when you make your sign charts!

1. $f(x) = \dfrac{1}{x}$
2. $f(x) = \dfrac{x^2}{x^2 - 4}$

   To help you out, the first two derivatives are

$$f'(x) = -\frac{8x}{(x^2-4)^2}, \quad f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}.$$

Solutions

Problem 1

We are given that $f(x) = \frac{1}{x}$, which is ◯6. Using the Power Rule, the first two derivatives are:

$$f'(x) = -x^{-2} = -\frac{1}{x^2}, \quad f''(x) = -(-2)x^{-3} = \frac{2}{x^3}.$$

1. Asymptotes. The degree of the numerator is $N = 0$ and the degree of the denominator is $D = 1$. Since $N < D$, we see that there is a horizontal asymptote at $y = 0$.

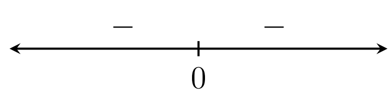
   For vertical asymptotes, we set the denominator equal to 0. So $x = 0$ is a vertical asymptote. Looking at the graph, we describe the behavior of the function at the asymptotes using limit notation:

   $$\lim_{x\to 0^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x\to 0^+} f(x) \text{ DNE } (+\infty).$$

2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0. But the numerator is 1, and so can never be 0. So there are no local minima and maxima.

3. Intervals of Increase and Decrease. To make a sign chart, we need to use the points where $f'(x) = 0$ or where the function is undefined. Thus, since $f'(x)$ is never 0, we just use $x = 0$, the vertical asymptote. Easy test points are $x = -1$ and $x = 1$. Evaluating:

   $$f'(-1) = -1, \quad f'(1) = -1.$$

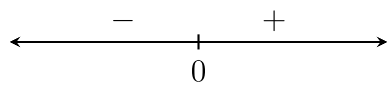
   

   Reading off the sign chart, we see that $f(x)$ is decreasing when on $(-\infty, 0)$ and $(0, \infty)$, since $f'(x) < 0$ there. Note that we cannot write this as $(-\infty, \infty)$, since this interval includes 0, but the function is not defined there.

4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0. But the numerator of $f''(x)$ is 2, and so can never be 0. Thus, there are no inflection points.

5. Intervals of Concavity. There are no points where $f''(x) = 0$, so we use only $x = 0$, which corresponds to the vertical asymptote, to make our sign chart. Easy test values are -1 and 1. Evaluating:

$$f''(-1) = -2, \quad f''(1) = 2.$$



Thus, the graph of $f(x)$ is concave up on $(0, \infty)$ since $f''(x) > 0$ there, and concave down on $(-\infty, 0)$ since $f''(x) < 0$ there.

Problem 2

We are given that $f(x) = \dfrac{x^2}{x^2 - 4}$, which is ◯7. The first two derivatives are:

$$f'(x) = -\frac{8x}{(x^2-4)^2}, \quad f''(x) = \frac{8(3x^2+4)}{(x^2-4)^3}.$$

1. Asymptotes. The degree of the numerator is $N = 2$ and the degree of the denominator is $D = 2$. Since $N = D$, we take the ratio of the leading coefficients of the numerator and denominator – both 1 in this case – to get the horizontal asymptote $y = \dfrac{1}{1} = 1$.

   For vertical asymptotes, we set the denominator equal to 0.

   $$\begin{aligned} x^2 - 4 &= 0 \\ (x+2)(x-2) &= 0 \\ x &= -2, 2 \end{aligned}$$

   So there are vertical asymptotes at $x = -2$ and $x = 2$. Looking at the graph, we describe the behavior of the function at these asymptotes using limit notation:

   $$\lim_{x \to -2^-} f(x) \text{ DNE } (+\infty), \quad \lim_{x \to -2^+} f(x) \text{ DNE } (-\infty),$$

   $$\lim_{x \to 2^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x \to 2^+} f(x) \text{ DNE } (+\infty).$$

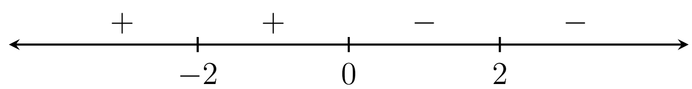
2. Local Minima and Maxima. We need to solve $f'(x) = 0$, so we set the numerator of $f'(x)$ equal to 0.

   $$\begin{aligned} -8x &= 0 \\ x &= 0 \end{aligned}$$

   Since $f''(0) = -\dfrac{1}{2}$, the graph is concave down at $x = 0$. Thus, there is a local maximum at $(0, 0)$.

3. Intervals of Increase and Decrease. To make a sign chart, we need to use the points where $f'(x) = 0$ or where the function is undefined. Thus, we use $x = 0$, (just calculated), and the vertical asymptotes -2 and 2. Easy test points are -3, -1, 1, and 3. Evaluating:

$$f'(-3) = \frac{24}{25}, \quad f'(-1) = \frac{8}{9}, \quad f'(1) = -\frac{8}{9}, \quad f'(3) = -\frac{24}{25}$$



Reading off the sign chart, we see that $f(x)$ is increasing when on $(-\infty, -2)$ and $(-2, 0)$, since $f'(x) > 0$ there. Note that we cannot write this as $(-\infty, 0)$, since this interval includes -2, but the function is not defined there. The function is decreasing on $(0, 2)$ and $(2, \infty)$ since $f'(x) < 0$ there. Again, two separate intervals are needed.

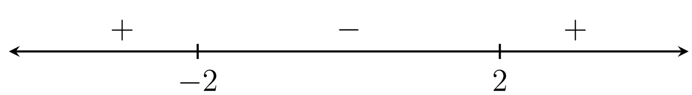
4. Inflection Points. To find possible inflection points, we set the numerator of $f''(x)$ equal to 0.

$$\begin{aligned} 8x(3^2 + 4) &= 0 \\ 3x^2 + 4 &= 0 \\ 3x^2 &= -4 \\ x^2 &= -\frac{4}{3} && \text{impossible} \end{aligned}$$

Since x^2 can never be negative, there are no inflection points.

5. Intervals of Concavity. There are no points where $f''(x) = 0$, so we use only -2 and 2, which correspond to the vertical asymptotes, to make our sign chart. Easy test values are -3, 0, and 3. Evaluating:

$$f''(-3) = \frac{248}{125}, \quad f''(0) = -\frac{1}{2}, \quad f''(3) = \frac{248}{125}.$$



Thus, the graph of $f(x)$ is concave up on $(-\infty, -2)$ and $(2, \infty)$ since $f''(x) > 0$ there, and concave down on $(-2, 2)$ since $f''(x) < 0$ there.