Summary of Continuity and Differentiation

As we did with limits, we'll now summarize where continuity and differentiation are important in calculus.

Continuity

1. **Graphs.** Continuity is helpful in decribing features of graphs. If a is in the domain of a function $f(x)$, we say that

   (a) $f(x)$ has a removable discontinuity at a if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal, but $f(a)$ is *not* equal to this value.

   (b) $f(x)$ has an essential discontinuity at a if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, but are not equal to each other.

   (c) $f(x)$ is continuous at a if both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal, and $f(a)$ *is* equal to this value.

   (d) The function $f(x)$ is continuous if it is continuous at all points a in the domain.

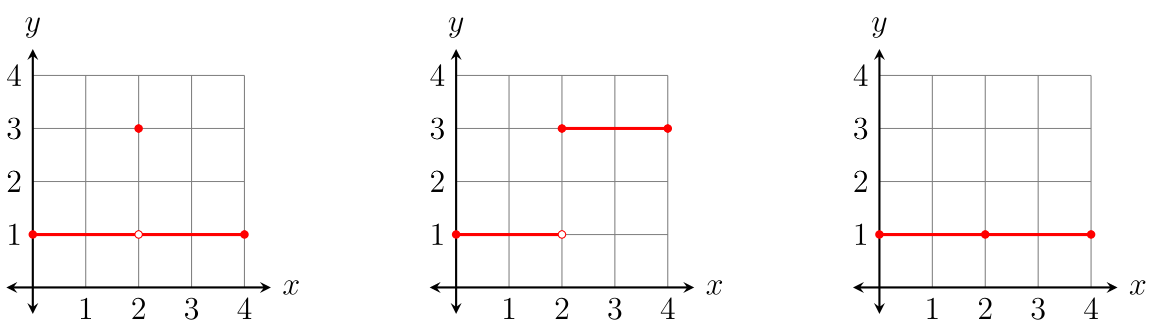


Figure 1: At $x = 2$: Removable discontinuity (left), essential discontinuity (middle), continuous (right).

See Day 17 on the course website.

2. **Intermediate Value Theorem (IVT).** The Intermediate Value Theorem is usually stated as follows.

> Suppose $f(x)$ is a continuous function defined on a closed interval $[a, b]$. If $f(a) \neq f(b)$, and if c is between $f(a)$ and $f(b)$, then there is some x_0 in the open interval (a, b) such that $f(x_0) = c$.

We applied this by showing that two curves must intersect. The geometry is this: if the blue curve $f(x)$ is above the red curve $g(x)$ at one endpoint of a closed interval, and the red curve is above the blue curve at the other endpoint, they have to cross somewhere in middle, assuming the curves are continuous.

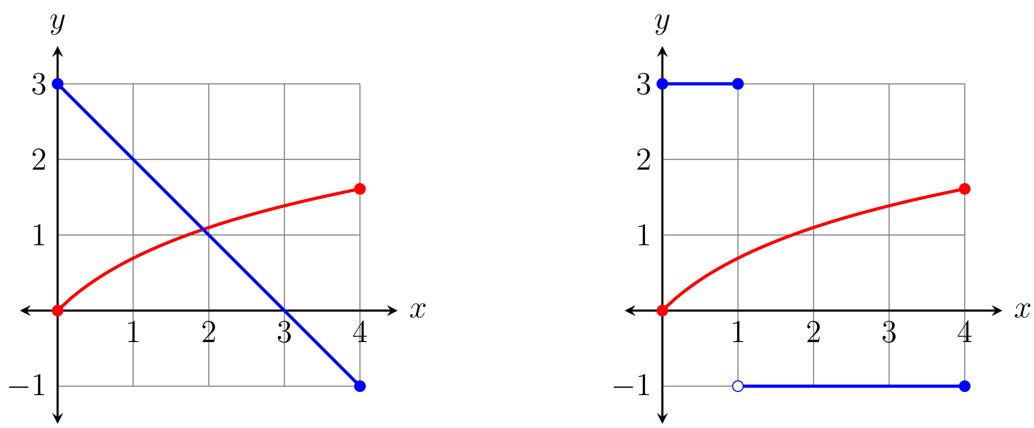


Figure 2: Continuous curves (left), at least one curve not continuous (right).

On the right in Figure 2, you can see that if the blue curve is above the red curve on the left, but below on the right, and one of the curves is *not* continuous, the curves do not have to intersect. But if both curves are continuous, they *must* intersect.

How do we use the IVT to show this? Given the geometry of the curves, the function $f(x) - g(x)$ must be negative at one endpoint and positive at the other. Since 0 lies between any negative and positive number, there is a point x_0 in the interval where $f(x_0) - g(x_0) = 0$, which means $f(x_0) = g(x_0)$. Therefore the curves intersect at x_0.

Details may be found at Day 20 on the course website.

3. **Extreme Value Theorem (EVT).** The Extreme Value Theorem states:

> If a function is defined on a closed interval and is continuous, both a global minimum and a global maximum exist.

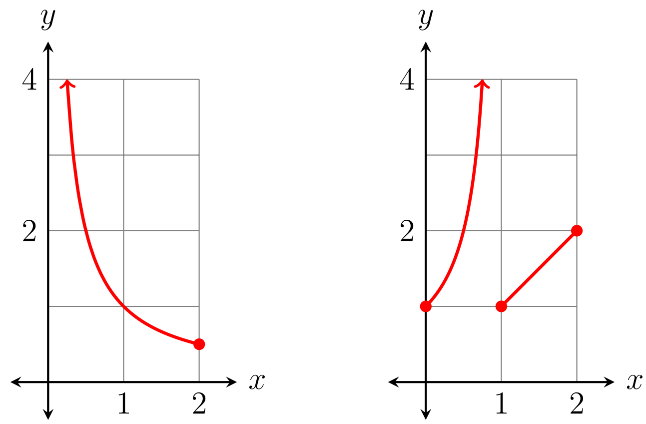


Figure 3: The importance of a closed interval (left), and continuiuty (right).

It is important that the interval is closed, since on the left of Figure 3, if the domain is $(0, 2]$, you can have a vertical asymptote. And if the function is not continuous, you might have a vertical asymptote inside the interval, as shown on the right of Figure 3.

How do you find global extrema? You have to look at values of the function where $f'(x) = 0$ or where the derivative is undefined, and also at the endpoints. So this theorem involves both continuity and differentiation.

See Days 18 and 19 on the course website.

Differentiation

4. **Finding rates of change.** One important use of the derivative in all sciences is to find rates of change. We looked at a few primary examples. If you had a function of displacement in km as a function of time in hours, you would find the rate of change – which is just the velocity – in units of km/hr. You find the rate of change by taking the derivative of the function.

   Another example we looked is exponential growth of organisms. Bacteria in a Petri dish start to grow exponentially, but as the dish gets full, the growth rate slows down. But an exponential function is a good model for what happens at the beginning.

   The variable P (for population) is often used to describe exponential growth. When you take $P'(t)$, where t is in hours, you are finding a rate of change. That is, you're looking at approximately how many bacteria per hour the population is growing. Again, you do this by taking the derivative and then plugging in your given value of t.

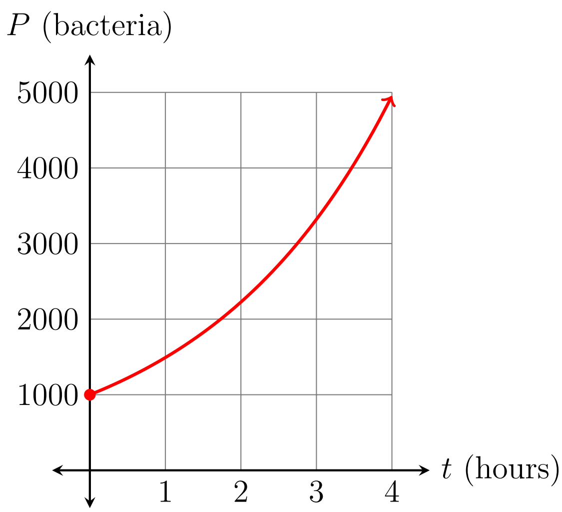


Figure 4: Graph of $P(t) = 1000e^{0.4t}$.

   More about exponential functions may be found on Day 12 of the course website.

5. **Finding equations of tangent lines.** The rate of change is *also* the slope of a tangent line to the graph of a function. Given a point on a graph, we can use the derivative to find the slope of the tangent line, and then find an equation for the tangent line.

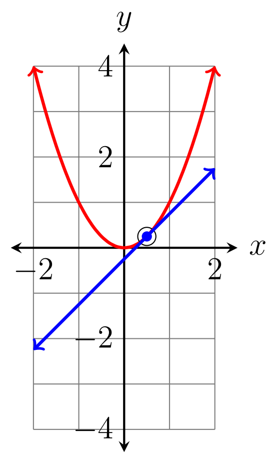


Figure 5: Tangent line on a graph.

See Day 6 for examples of finding equations of tangent lines.

6. **Find where a function is Increasing/decreasing.** The derivative is also useful to find out where a function is increasing or decreasing. When $f'(x) > 0$, the function is increasing, and when $f'(x) < 0$, the function is decreasing (see Figure 6 (left)). When $f'(x) = 0$, more work has to be done. In this case, there could be a local extremum, or the function could also be decreasing or increasing (as in Figure 6 (right)).

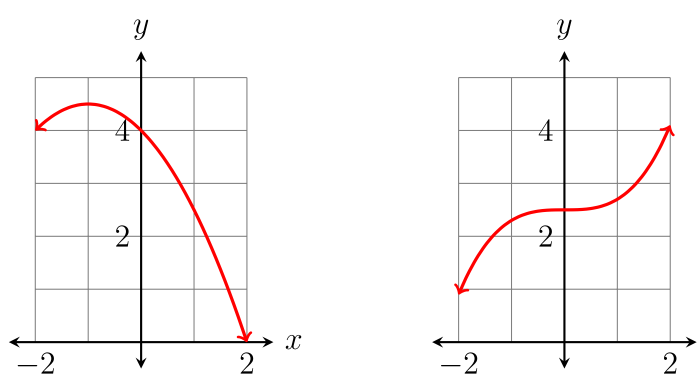


Figure 6: Increasing on $(-2, -1)$ and decreasing on $(-1, 2)$ (left), and increasing when $f'(0) = 0$ (right).

See Day 6 for a discussion of these ideas.

7. **Determining concavity and finding points of inflection.** Here, we need the second derivative. When $f''(x) > 0$, the graph is concave up (as in the left of Figure 7). When $f''(x) < 0$, the graph is concave down (see the middle of Figure 7). We do check when $f''(x) = 0$ to find inflection points, but more work is needed because there can also be a local minimum or maximum when $f''(x) = 0$ (as in the right of Figure 7). In this case, we either need a graph, or if we don't have one, making a sign chart is necessary.

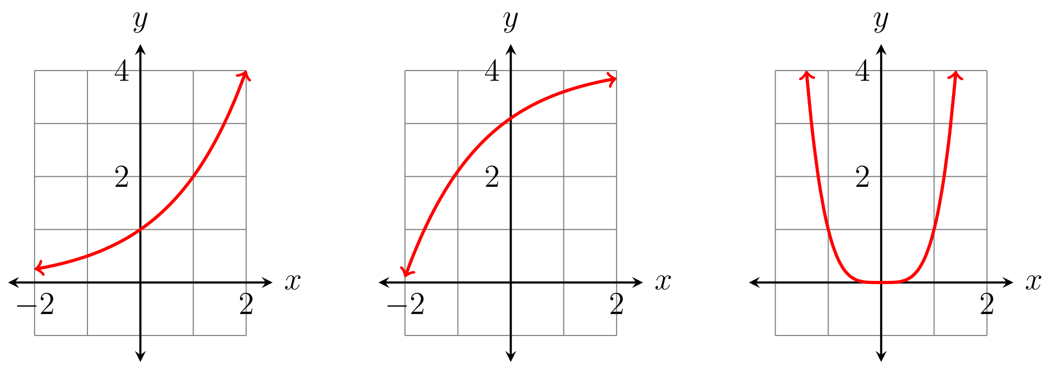


Figure 7: Graph of $f''(x) > 0$ (left), $f''(x) < 0$ (middle), and $f''(0) = 0$ (right).

See Day 11 on the course website.

8. **Finding local and global extrema.** We find local extrema by solving $f'(x) = 0$ or seeing where $f'(x)$ does not exist. When $f'(x) = 0$, we can use the second derivative to see if the extremum is a minimum of maximum. If $f''(x) > 0$, the graph is concave up, and so it is a local minimum (see the left of Figure 8). If $f''(x) < 0$, the graph is concave down, and so it is a local maximum (middle of Figure 8). When $f''(x) = 0$, we need to make a sign chart, since it is possible there could be a local minimum (as in the right of Figure 8), a local maximum, or an inflection point.

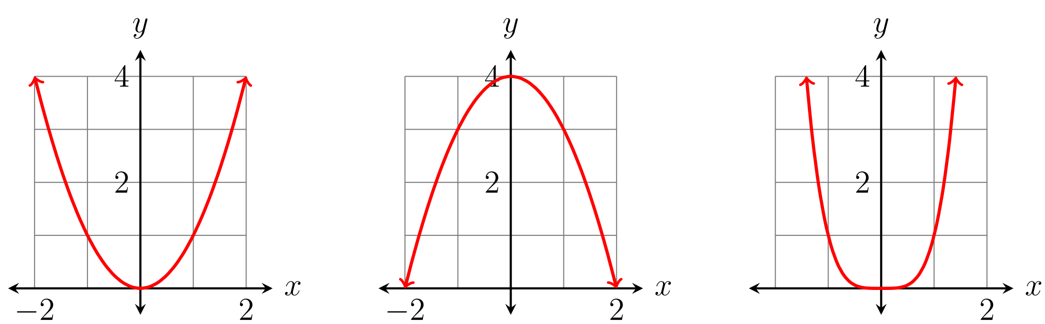


Figure 8: Graph of $f''(0) > 0$ (left), $f''(0) < 0$ (middle), and $f''(0) = 0$ (right).

See Day 18 for more discussion about local extrema.

See the Continuity section for a discussion of global extrema.

9. **L'Hôpital's Rule.** Differentiation is needed to use L'Hôpital's Rule, which is used when limits are of the form $\dfrac{\pm\infty}{\pm\infty}$, $\dfrac{0}{0}$, or $\pm\infty \cdot 0$. Suppose that $f(x)$ and $g(x)$ are functions, and a is either a real number or $\pm\infty$. Then

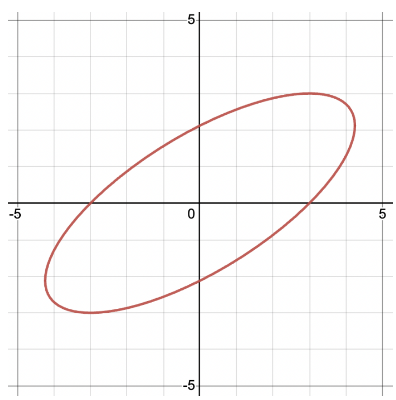
   (a) If $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) as $x \to a$, L'Hôpital's Rule says that
   $$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{\text{LR}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

   (b) If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, L'Hôpital's Rule says that
   $$\lim_{x\to a} \frac{f(x)}{g(x)} \stackrel{\text{LR}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

   (c) If one of $f(x)$ and $g(x)$ goes to 0 and the other goes to $\pm\infty$ as $x \to a$, you must rewrite by moving one of the function to the denominator and then applying L'Hôpital's Rule.

   See Day 22 for a discussion of (a) and (b), and see Day 25 for a discussion of (c).

10. **Finding tangents to general curves.** Many curves – like circles, ellipses, and hyperbolas are *not* graphs of functions because they fail the vertical line test. One such example is the ellipse $x^2 - 2xy + 2y^2 = 9$, shown below.



   In this case, we do not have a function $y = f(x)$, but rather we say that y is defined *implicitly*. In such cases, we can use implicit differentiation to find $\dfrac{dy}{dx}$. Once we find $\dfrac{dy}{dx}$, we can use the derivative to find tangent lines, asymptotes, etc.
   See Day 26 for examples of how implicit differentiation is used.

Finding Derivatives

Here is a summary of all the derivatives we know (that is, you can just use them at any time without justification), and the basic rules of differentiation.

1. $\dfrac{d}{dx}\sin(x) = \cos(x)$.
2. $\dfrac{d}{dx}\cos(x) = -\sin(x)$.
3. $\dfrac{d}{dx}\tan(x) = \sec^2(x)$.
4. $\dfrac{d}{dx}e^x = e^x$.
5. $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.
6. $\dfrac{d}{dx}b^x = b^x \ln b$.
7. $\dfrac{d}{dx}\log_b(x) = \dfrac{1}{x\ln b}$.
8. $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$.
9. $\dfrac{d}{dx}\arccos(x) = -\dfrac{1}{\sqrt{1-x^2}}$.
10. $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{x^2+1}$.
11. The Power Rule: When $n > 0$,
$$\frac{d}{dx}x^n = nx^{n-1}.$$
12. The Sum Rule:
$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x).$$
13. The Difference Rule:
$$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x).$$
14. The Constant Multiple Rule:
$$\frac{d}{dx}(cf(x)) = cf'(x).$$
15. The Product Rule:
$$\frac{d}{dx}f(x)g(x) = f(x)g'(x)+g(x)f'(x).$$
16. The Quotient Rule:
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}.$$
17. The Chain Rule:
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$