Inverse Trigonometry II

Derivative of $\arcsin(x)$.

Now that we have a good understanding of the inverse trigonometric functions, it's time to look at their derivatives. We'll be able to make some headway using implicit differentiation. Let's start by writing $y = \arcsin(x)$ as $x = \sin(y)$, always assuming that y is in the range of $\arcsin(x)$. Now use implicit differentiation:

$$\begin{aligned} x &= \sin(y) \\ \frac{d}{dx}x &= \frac{d}{dx}\sin(y) \\ 1 &= \cos(y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} \end{aligned}$$

Now the question is: what do we do with $\cos(y)$? Since $\arcsin(x)$ is a function of x, our derivative should also be a function of x. Here, we use one of the Pythagorean Identities from trigonometry: for any θ, $\sin^2(\theta) + \cos^2(\theta) = 1$. We'll substitute y in for θ and solve.

$$\begin{aligned} \sin^2(y) + \cos^2(y) &= 1 \\ \cos^2(y) &= 1 - \sin^2(y) \\ \cos^2(y) &= 1 - x^2 && \text{since } x = \sin(y) \\ \cos(y) &= \sqrt{1 - x^2} \\ \frac{dy}{dx} &= \frac{1}{\cos(y)} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

We do have address the fact that we solved for $\cos(y)$ as $+\sqrt{1 - x^2}$ instead of $-\sqrt{1 - x^2}$. Recall that the graph of $\arcsin(x)$ is always increasing, which means its derivative always has to be positive. That's why we could take the positive square root.

Derivative of $\arccos(x)$.

We can use the same approach as we did for $\arcsin(x)$. But there is an easier way if we look at the right triangle in Figure 1.

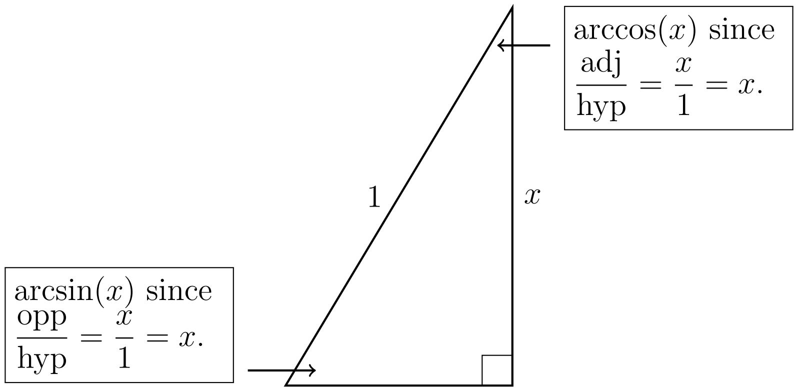

Figure 1: Showing $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$.

The angle at the upper right is $\arccos(x)$ because the cosine is $\frac{\text{adj}}{\text{hyp}}$, and the adjacent side relative to this angle is x and the hypotenuse is 1. Similarly, the angle at the bottom left is $\arcsin(x)$ because the sine is $\frac{\text{opp}}{\text{hyp}}$, and the opposite side relative to this angle is x and the hypotenuse is 1.

Since the angles of a triangle add up to 180°, and since there's already a right angle, which is 90°, the other two angles must add up to 90°. But in Calculus, we always use radians, and so

$$\arcsin(x) + \arccos(x) = \frac{\pi}{2}.$$

This helps because we can solve for $\arccos(x)$ and then use what we just learned.

$$\begin{aligned}
\arcsin(x) + \arccos(x) &= \frac{\pi}{2} \\
\arccos(x) &= \frac{\pi}{2} - \arcsin(x) \\
\frac{d}{dx}\arccos(x) &= \frac{d}{dx}\frac{\pi}{2} - \frac{d}{dx}\arcsin(x) \\
&= 0 - \frac{1}{\sqrt{1-x^2}} \\
&= -\frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

13 **Derivative of** $\arctan(x)$.

While we know that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, we do not have a similar formula for $\arctan(x)$. So we have to use implicit differentiation again. We write $x = \tan(y)$, again assuming that y is in the range of $\arctan(x)$.

$$\begin{aligned} x &= \tan(y) \\ \frac{d}{dx}x &= \frac{d}{dx}\tan(y) \\ 1 &= \sec^2(y)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2(y)} \end{aligned}$$

Again, we need to change the y to x. We can start with the identity we used before, and divide by $\cos^2(\theta)$.

$$\begin{aligned} \sin^2(\theta) + \cos^2(\theta) &= 1 \\ \frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} &= \frac{1}{\cos^2(\theta)} \\ \tan^2(\theta) + 1 &= \sec^2(\theta) \end{aligned}$$

So let's substitute this back in and simplify, using y instead of θ.

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\sec^2(y)} \\ &= \frac{1}{\tan^2(y) + 1} \\ &= \frac{1}{x^2 + 1} && \text{since } x = \tan(y) \end{aligned}$$

14 ## Examples

15 1. Find $h'(x)$ if $h(x) = x\arctan(x)$.

   Here, we need the product rule with $f(x) = x$ and $g(x) = \arctan(x)$.

$$\begin{aligned} h(x) &= x\arctan(x) \\ h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x \cdot \frac{1}{x^2+1} + \arctan(x) \cdot 1 \\ &= \frac{x}{x^2+1} + \arctan(x). \end{aligned}$$

16 2. Find $h'(x)$ if $h(x) = \arccos(x^2)$.

   Use the Chain Rule, with $f(x) = \arccos(x)$ and $g(x) = x^2$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= -\frac{1}{\sqrt{1-(g(x))^2}} \cdot 2x \\ &= -\frac{2x}{\sqrt{1-(x^2)^2}} \\ &= -\frac{2x}{\sqrt{1-x^4}} \end{aligned}$$

17 3. Find $h'(x)$ if $h(x) = \arcsin(2x-1)$. Simplify.

   Use the Chain Rule, with $f(x) = \arcsin(x)$ and $g(x) = 2x - 1$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{\sqrt{1-(g(x))^2}} \cdot 2 \\ &= \frac{2}{\sqrt{1-(2x-1)^2}} \\ &= \frac{2}{\sqrt{1-(4x^2-4x+1)}} \\ &= \frac{2}{\sqrt{4x-4x^2}} \\ &= \frac{2}{\sqrt{4}\sqrt{x-x^2}} \\ &= \frac{1}{\sqrt{x-x^2}} \end{aligned}$$

Homework

1. If $h(x) = x^2 \arcsin(x)$, find $h'(x)$.
2. If $h(x) = \arctan(2x + 1)$, find $h'(x)$.
3. If $h(x) = \arccos(1 - x)$, find $h'(x)$.
4. If $\arcsin(y) + y = x$, find $\dfrac{dy}{dx}$.
5. Note that $\lim_{x\to\infty} \dfrac{1}{x^2 + 1} = 0$. How does this relate to the horizontal asymptotes of the graph of $y = \arctan(x)$?
6. Find an equation of the tangent line to $\arccos(x)$ at $x = \dfrac{\sqrt{3}}{2}$. Check with desmos. If you type "y=arccos(x)" into desmos, you'll get the graph of $y = \arccos(x)$.

Solutions

1. Use the Product Rule, with $f(x) = x^2$ and $g(x) = \arcsin(x)$.

$$\begin{aligned} h(x) &= f(x)g(x) \\ h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= x^2 \cdot \frac{1}{\sqrt{1-x^2}} + \arcsin(x) \cdot 2x \\ &= \frac{x^2}{\sqrt{1-x^2}} + 2x\arcsin(x) \end{aligned}$$

2. Use the Chain Rule, with $f(x) = \arctan(x)$ and $g(x) = 2x + 1$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{(g(x))^2+1} \cdot 2 \\ &= \frac{2}{(2x+1)^2+1} \\ &= \frac{2}{(4x^2+4x+1)+1} \\ &= \frac{2}{4x^2+4x+2} \\ &= \frac{2}{2(2x^2+2x+1)} \\ &= \frac{1}{2x^2+2x+1} \end{aligned}$$

3. Use the Chain Rule, with $f(x) = \arccos(x)$ and $g(x) = 1 - x$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= -\frac{1}{\sqrt{1-(g(x))^2}} \cdot (-1) \\ &= \frac{1}{\sqrt{1-(1-x)^2}} \\ &= \frac{1}{\sqrt{1-(1-2x+x^2)}} \\ &= \frac{1}{\sqrt{2x-x^2}} \end{aligned}$$

4. For this problem, we need to use implicit differentiation. After Step 3, we multiply through by $\sqrt{1-y^2}$ to eliminate fractions to make the algebra easier.

$$\begin{aligned}
\arcsin(y) + y &= x \\
\frac{d}{dx}\arcsin(y) + \frac{d}{dx}y &= \frac{d}{dx}x \\
\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} + \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} + \sqrt{1-y^2}\frac{dy}{dx} &= \sqrt{1-y^2} \\
(1+\sqrt{1-y^2})\frac{dy}{dx} &= \sqrt{1-y^2} \\
\frac{dy}{dx} &= \frac{\sqrt{1-y^2}}{1+\sqrt{1-y^2}}
\end{aligned}$$

5. We know that $y = \arctan(x)$ has a horizontal asymptote at $y = \dfrac{\pi}{2}$. This means that as $x \to \infty$, the curve has to flatten out in order to approach the asymptote without crossing it. So as the curve flattens out, the slope of the tangent line gets closer and closer to 0.

6. First, we find the slope of the tangent line by plugging in $\dfrac{\sqrt{3}}{2}$ into the derivative.

$$\begin{aligned}
m &= -\frac{1}{\sqrt{1-(\sqrt{3}/2)^2}} \\
&= -\frac{1}{\sqrt{1-3/4}} \\
&= -\frac{1}{\sqrt{1/4}} \\
&= -\frac{1}{1/2} \\
&= -2
\end{aligned}$$

Since $\arccos(\sqrt{3}/2) = \pi/6$, we use the point $(\sqrt{3}/2, \pi/6)$.

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - \pi/6 &= -2(x - \sqrt{3}/2) \\
y - \pi/6 &= -2x + \sqrt{3} \\
y &= -2x + \sqrt{3} + \pi/6
\end{aligned}$$