Inverse Trigonometry I

The most famous pair of inverse functions in calculus is e^x and $\ln x$. We learned a lot about $\ln x$ by reflecting e^x across the line $y = x$. Also, it was very important that when we reflected across the line $y = x$, the graph passed the vertical line test, so we were able to define the *function* $f(x) = \ln x$. Each x corresponded to exactly one y.

To follow along, you will need to visit desmos.com. Trigonometric functions are also very important in calculus. But we can't just reflect along the line $y = x$ and be done with it. Let's see why. We'll start with $\sin(x)$, which you can see by selecting ◯1. Now select ◯2 and ◯3, and you'll see the graph of $\sin(x)$ reflected along the line $y = x$. Notice we literally switch the x and y from $y = \sin(x)$ to $x = \sin(y)$ to see the reflection.

Now select ◯4. You'll notice that $x = \sin(y)$ does *not* pass the vertical line test, and so it is not a function. How can we create a function?

If you select ◯5, you see a small part of the graph of $x = \sin(y)$. This part *does* pass the vertical line test, and it is this part of the curve that we use to define the inverse function, $\arcsin(x)$. Many books write $\sin^{-1}(x)$ for the inverse function, but this is confusing since you might think $\sin^{-1}(x) = \dfrac{1}{\sin(x)}$. When you use $\arcsin(x)$, there is no confusion. Just note this in case you look at online resources.

One big difference here. Since e^x and $\ln x$ are inverse functions, $y = e^x$ means exactly the same thing as $x = \ln y$. They are inverses of each other. But

If $y = \sin(x)$, then it DOES NOT ALWAYS MEAN THAT $x = \arcsin(y)$.

This fact is what makes working with inverse trigonometric functions challenging. Consider e^x and $\ln x$ again. Using interval notation, the domain of e^x is $(-\infty, \infty)$ and the range is $(0, \infty)$. The domain of $\ln(x)$ is $(0, \infty)$ while the range is $(-\infty, \infty)$. Here, the domain and range just switch.

But that can't happen with $\sin(x)$, because when you reflect across $y = x$, you *don't* get a function. Look back on desmos. Notice that the range of $\sin(x)$, $[-1, 1]$, is the domain of $\arcsin(x)$. But the domain of $\sin(x)$, which is $(-\infty, \infty)$, is *not* the range of $\arcsin(x)$, otherwise the vertical line test would fail. So the range of $\arcsin(x)$ is $[-\pi/2, \pi/2]$, since if the range were made any larger, the graph would fail the vertical line test.

Now select only ◯1 and ◯6. When you deselect ◯1, you'll notice that only one piece of $\sin(x)$ remains. This is called **restricting the domain.** Now select ◯2 and ◯5 again. When you reflect $y = \sin(x)$ with restricted domain, you get a function. So that means:

If $y = \sin(x)$, and if x is in the restricted domain $[-\pi/2, \pi/2]$, then $x = \arcsin(y)$.

Another way of saying it is this. The domain of the restricted $\sin(x)$, which is $[-\pi/2, \pi/2]$, is the range of $\arcsin(x)$. The range of the restricted $\sin(x)$, which is $[-1, 1]$, is the same as the domain of $\arcsin(x)$.

Also note that this is exactly how we defined $\sqrt{x}$. We had to restrict the domain of $y = x^2$ to $[0, \infty)$ in order to get the inverse function. But we are so familiar with the square root function, we hardly notice. Inverse trigonometric functions are not so familiar.

So, because 0, $\pi/4$, and $-\pi/3$ are all in the range of $\arcsin(x)$, then

$$\arcsin(\sin(0)) = 0, \quad \arcsin(\sin(\pi/4)) = \pi/4, \quad \arcsin(\sin(-\pi/3)) = -\pi/3.$$

But because π and $2\pi/3$ are not in the range of $\arcsin(x)$, then

$$\arcsin(\sin(\pi)) \neq \pi, \quad \arcsin(\sin(2\pi/3)) \neq 2\pi/3.$$

So we need a way to work these out. Who comes to the rescue? The unit circle, of course.

Example 1: $\arcsin(\sin(x))$.

Sometimes it's the case that $\arcsin(\sin(x)) \neq x$. $\arcsin(\sin(2\pi/3)) \neq 2\pi/3$, since $2\pi/3$ is not in the range of $\arcsin(x)$. So how do we go about finding $\arcsin(\sin(2\pi/3))$?

Let's start with a unit circle.

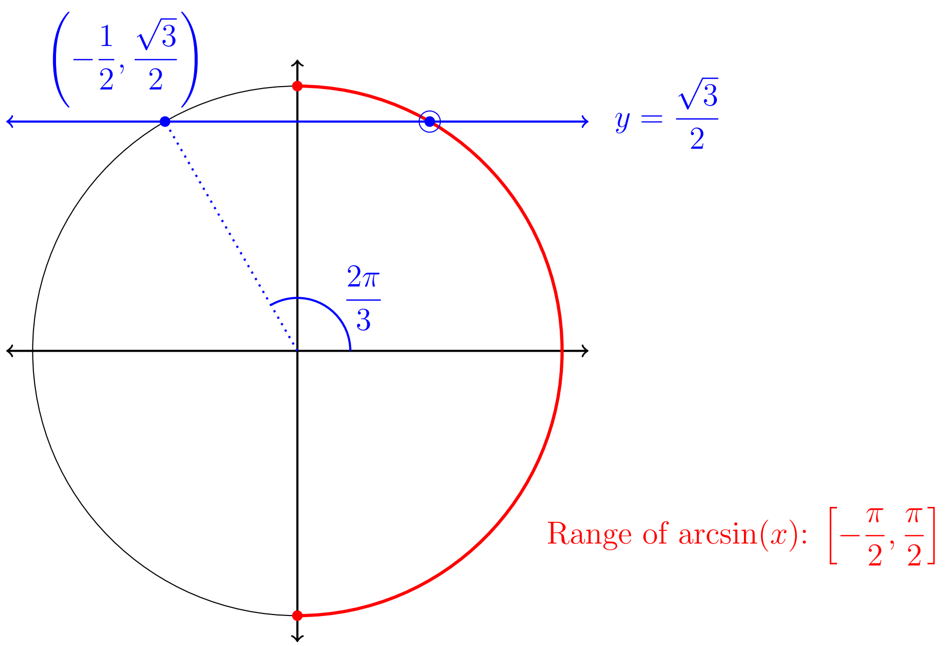


Figure 1: Calculating $\arcsin(\sin(2\pi/3))$.

We need to find the appropriate angle in the range of $\arcsin(x)$ whose sine is the same as the sine of $2\pi/3$.

1. Draw a unit circle, and highlight (here in red) the range of $\arcsin(x)$.
2. Since we're looking for $\arcsin(\sin(2\pi/3))$, find the point on the unit circle corresponding to $2\pi/3$ and label the coordinates (blue dot on the left of Figure 8).
3. Since $\sin(x)$ is the y-coordinate on the unit circle, draw a horizontal line through this point until in intersects the range of $\arcsin(x)$ (circled blue dot on the right).
4. Find which angle in the range of $\arcsin(x)$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, corresponds to this point on the unit circle.
5. Since $\sin(\pi/3) = \sqrt{3}/2$, then $\arcsin(\sin(2\pi/3)) = \pi/3$.

To summarize, we are essentially asking the question, "What angle in the range of $\arcsin(x)$ has the same sine as $2\pi/3$?"

What about inverses of $\cos(x)$ and $\tan(x)$? We won't go into all the details here, since the basic concept is the same: restrict the domain so that when you reflect the graph, you get the graph of a function – that is, you pass the vertical line test. Select ◯2 and ◯7. When you reflect over $y = x$, you get ◯8. If you select ◯4 again, you'll quickly notice that $x = \cos(y)$ does *not* pass the vertical line test. So, we restrict the domain of $\cos(x)$ to $[0, \pi]$. When you reflect $y = \cos(x)$ with this restricted domain, you get ◯9. See this by selecting ◯2, ◯7, ◯9, and ◯10 only. When you deselect ◯7, you'll see *only* that part of $y \cos(x)$ with domain $[0, \pi]$. Then the inverse relationship is clear. This means that $\arccos(x)$ is the inverse of $y = \cos(x)$ with restricted domain $[0, \pi]$. Thus,

> If $y = \cos(x)$, and if x is in the restricted domain $[0, \pi]$, then $x = \arccos(y)$.

You'll see how to find $\arccos(\cos(x))$ when x does *not* belong to the restricted domain in Example 2. If x is in the restricted domain $[0, \pi]$, then it will always be the case that $\arccos(\cos(x)) = x$.

A similar thing happens with $\tan(x)$. You'll see if you take $y = \tan(x)$ by selecting ◯11, and reflecting about $y = x$ by selecting ◯2 and ◯12, the reflection does not pass the vertical line test. But if we restrict the domain to $(-\pi/2, \pi/2)$ (select ◯2 and ◯13 only) and reflect by selecting ◯14, the graph passes the vertical line test. It is important to note the parentheses: there are vertical asymptotes at $x = -\pi/2$ and $x = \pi/2$, since these points on the unit circle make vertical lines with the origin, and the slope of a vertical line is undefined.

Thus,

> If $y = \tan(x)$, and if x is in the restricted domain $(-\pi/2, \pi/2)$ then $x = \arctan(y)$.

In other words, $\arctan(\tan(x)) = x$ if x is in the restricted domain $(-\pi/2, \pi/2)$. We'll see in Example 3 how to handle the situation if x is not in the restricted domain.

Example 2: $\arccos(\cos(x))$.

Sometimes it's the case that $\arccos(\cos(x)) \neq x$. $\arccos(\cos(5\pi/4)) \neq 5\pi/4$, since $5\pi/4$ is not in the range of $\arccos(x)$. So how do we go about finding $\arccos(\cos(5\pi/4))$?

Again, we start with a unit circle.

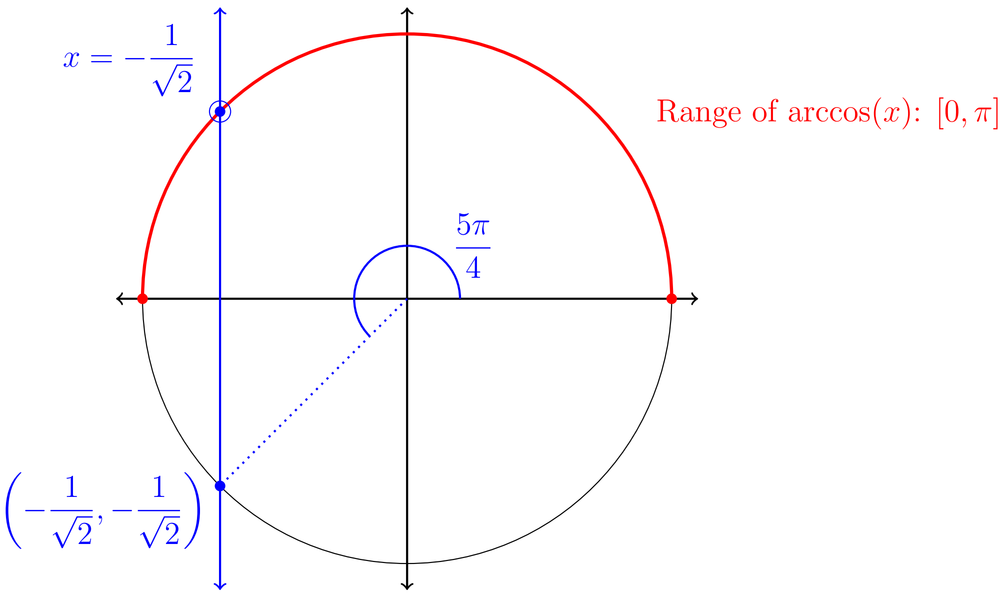


Figure 2: Calculating $\arccos(\cos(5\pi/4))$.

We need to find the appropriate angle in the range of $\arccos(x)$ whose cosine is the same as the cosine of $5\pi/4$.

1. Draw a unit circle, and highlight (here in red) the range of $\arccos(x)$.
2. Since we're looking for $\arccos(\cos(5\pi/4))$, find the point on the unit circle corresponding to $5\pi/4$ and label the coordinates (blue dot on the left of Figure 2).
3. Since $\cos(x)$ is the x-coordinate on the unit circle, draw a vertical line through this point until in intersects the range of $\arccos(x)$ (circled blue dot on the left).
4. Find which angle in the range of $\arccos(x)$, $[0, \pi]$, corresponds to this point on the unit circle.
5. Since $\cos(3\pi/4) = -1/\sqrt{2}$, then $\arccos(\cos(5\pi/4)) = 3\pi/4$.

To summarize, we are essentially asking the question, "What angle in the range of $\arccos(x)$ has the same cosine as $5\pi/4$?"

Example 3: $\arctan(\tan(x))$.

Sometimes it's the case that $\arctan(\tan(x)) \neq x$. $\arctan(\tan(5\pi/6)) \neq 5\pi/6$, since $5\pi/6$ is not in the range of $\arctan(x)$. So how do we go about finding $\arctan(\tan(5\pi/6))$?

Again, we start with a unit circle.

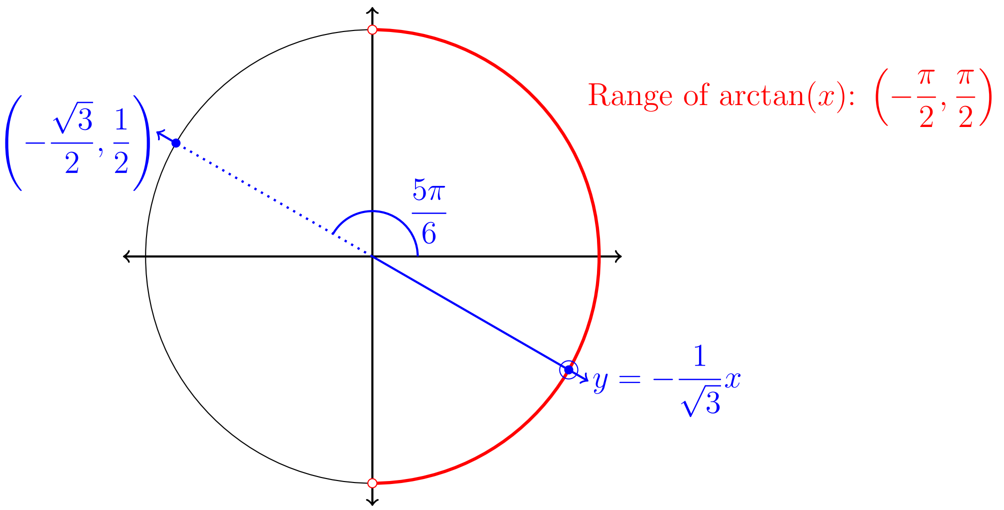


Figure 3: Calculating $\arctan(\tan(5\pi/6))$.

We need to find the appropriate angle in the range of $\arctan(x)$ whose tangent is the same as the tangent of $5\pi/6$.

1. Draw a unit circle, and highlight (here in red) the range of $\arctan(x)$.
2. Since we're looking for $\arctan(\tan(5\pi/6))$, find the point on the unit circle corresponding to $5\pi/6$ and label the coordinates (blue dot on the left of Figure 3).
3. Now
$$\tan(5\pi/6) = \frac{\sin(5\pi/6)}{\cos(5\pi/6)} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}}.$$
Since the sine corresponds to the y-coordinate and the cosine corresponds to the x-coordinate, then the tangent corresponds to $\frac{y}{x}$, which is the *slope* of the line through $(-\sqrt{3}/2, 1/2)$ and the origin. Draw this line, and see where it intersects the range of $\arctan(x)$ (circled blue dot on the right).
4. Find which angle in the range of $\arctan(x)$, $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, corresponds to this point on the unit circle.
5. Since $\tan(-\pi/6) = -1/\sqrt{3}$, then $\arctan(\tan(5\pi/6)) = -\pi/6$.

To summarize, we are essentially asking the question, "What angle in the range of $\arctan(x)$ has the same tangent as $5\pi/6$?"

So far, we've looked at how to evaluate $\arcsin(\sin(x))$, $\arccos(\cos(x))$, and $\arctan(\tan(x))$ for all x in the appropriate domain. What about the other way, that is, $\sin(\arcsin(x))$, $\cos(\arccos(x))$, and $\tan(\arctan(x))$? We saw that $\arcsin(\sin(2\pi/3)) \neq 2\pi/3$ because $2\pi/3$ is not in the range of $\arcsin(x)$.

Let's think about what $\sin(\arcsin(x))$ means. The domain of $\arcsin(x)$ is $[-1, 1]$. So x *must* be in the range of $\sin(x)$, because the range of $\sin(x)$ is *also* $[-1, 1]$. This means that $\sin(\arcsin(x)) = x$ for every x in the domain of $\arcsin(x)$, which is $[-1, 1]$. Said another way, any valid x you can plug into $\sin(\arcsin(x))$ will always be in the range of $\sin(x)$, and so $\sin(\arcsin(x)) = x$.

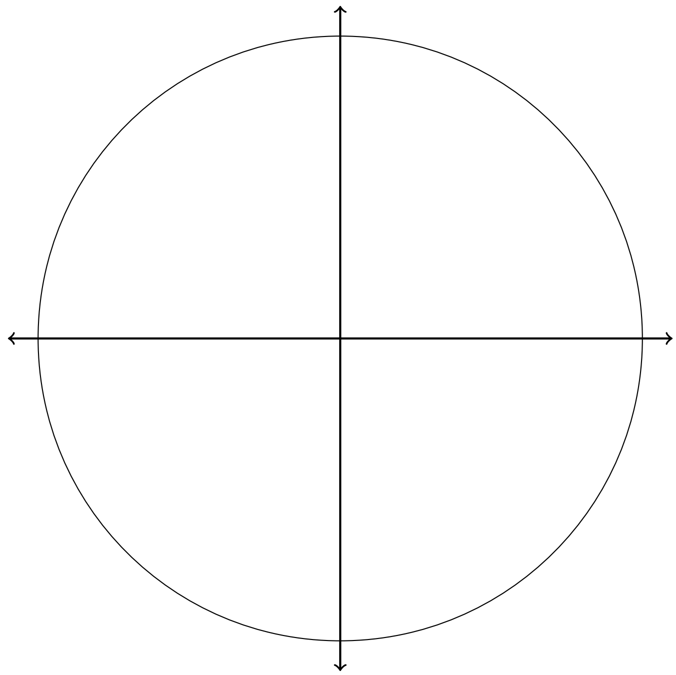
The exact same logic shows that $\cos(\arccos(x)) = x$ and $\tan(\arctan(x)) = x$ for all valid values of x.

The box below summarize all the important points. The tricky parts are 2(a), (b), and (c), where if x is not in the appropriate range, you have to work it out like Examples 1–3 above.

1. (a) For $y = \arcsin(x)$, the domain is $[-1, 1]$, range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
   (b) For $y = \arccos(x)$, the domain is $[-1, 1]$, and the range is $[0, \pi]$.
   (c) For $y = \arctan(x)$, the domain is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
2. (a) $\arcsin(\sin(x)) = x$ for all x in the range of $\arcsin(x)$.
   (b) $\arccos(\cos(x)) = x$ for all x in the range of $\arccos(x)$.
   (c) $\arctan(\tan(x)) = x$ for all x in the range of $\arctan(x)$.
3. (a) $\sin(\arcsin(x)) = x$ for all x in the domain of $\arcsin(x)$.
   (b) $\cos(\arccos(x)) = x$ for all x in the domain of $\arccos(x)$.
   (c) $\tan(\arctan(x)) = x$ for all x in the domain of $\arctan(x)$.

Homework

1. What is a restricted domain, and why is it necessary to define the inverse trigonometric functions?
2. Evaluate $\arccos(\cos(5\pi/3))$.
3. Evaluate $\sin(\arcsin(-\sqrt{3}/2))$.
4. Evaluate $\cos(\arccos(3/2))$.
5. Evaluate $\arctan(\tan(-\pi/4))$.
6. Evaluate $\arcsin(\sin(7\pi/4))$.
7. Evaluate $\tan(\arctan(-100))$.
8. Evaluate $\arccos(\cos(-\pi))$.
9. Evaluate $\arctan(\tan(5\pi/4))$.
10. Evaluate $\arcsin(\sin(4\pi/3))$.



Solutions

1. A restricted domain is when you restrict possible values for x. $\sin(x)$ is defined for all real numbers, but when using it to define $\arcsin(x)$, we restrict the domain to $[-\pi/2, \pi/2]$. We need to do this because when we reflect the graph of $\sin(x)$ across the line $y = x$, the graph does not pass the vertical line test.

2. $\arccos(\cos(5\pi/3)) = \pi/3$, as demonstrated below.

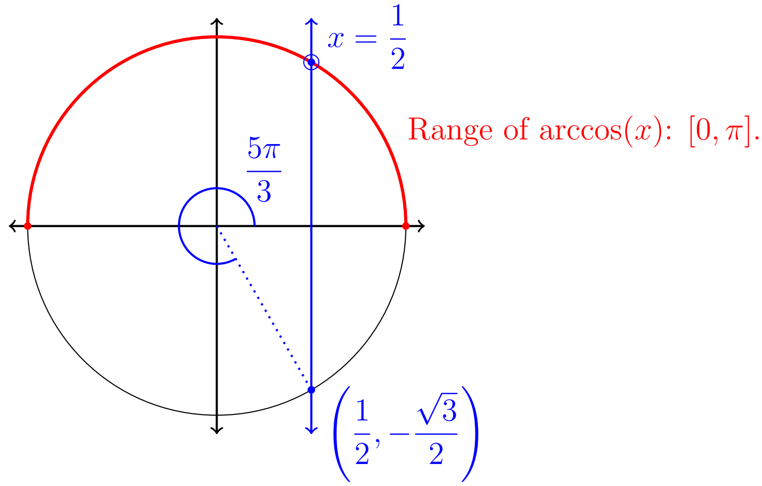


Figure 4: Calculating $\arccos(\cos(5\pi/3))$.

3. $\sin(\arcsin(-\sqrt{3}/2)) = -\sqrt{3}/2$, since $\sin(\arcsin(x)) = x$ for all x in the domain of $\arcsin(x)$.

4. $\cos(\arccos(3/2))$ is undefined because $3/2$ is not in the domain of $\arccos(x)$.

5. $\arctan(\tan(-\pi/4)) = -\pi/4$ because $-\pi/4$ is in the range of $\arctan(x)$.

6. $\arcsin(\sin(7\pi/4)) = -\pi/4$, as shown in the figure below. Note that $7\pi/4$ looks like it lies in the range of $\arcsin(x)$, but we must convert to an angle in $[-\pi/2, \pi/2]$, and so the answer is $-\pi/4$.

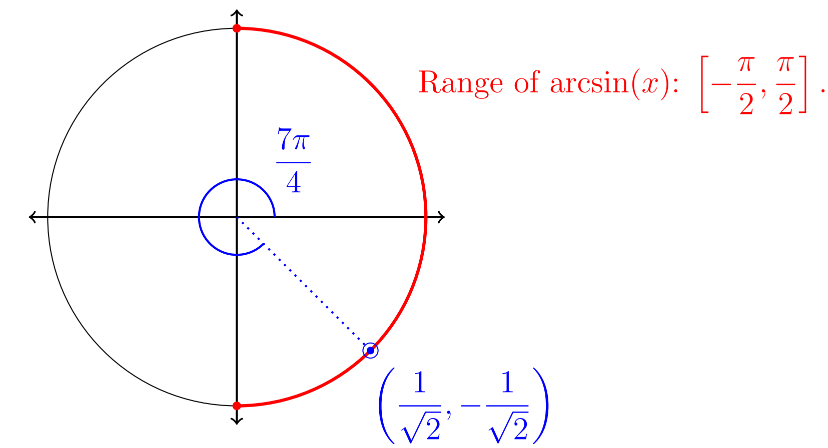


Figure 5: Calculating $\arcsin(\sin(7\pi/4))$.

7. $\tan(\arctan(-100)) = -100$, because $\tan(\arctan(x)) = x$ for all real numbers x.

8. $\arccos(\cos(-\pi)) = \pi$, as shown in the figure below. Note that it looks like $-\pi$ is in the range of $\arccos(x)$, but we must convert to an angle in the range of $\arccos(x)$, which is $[0, \pi]$. So the answer is π.

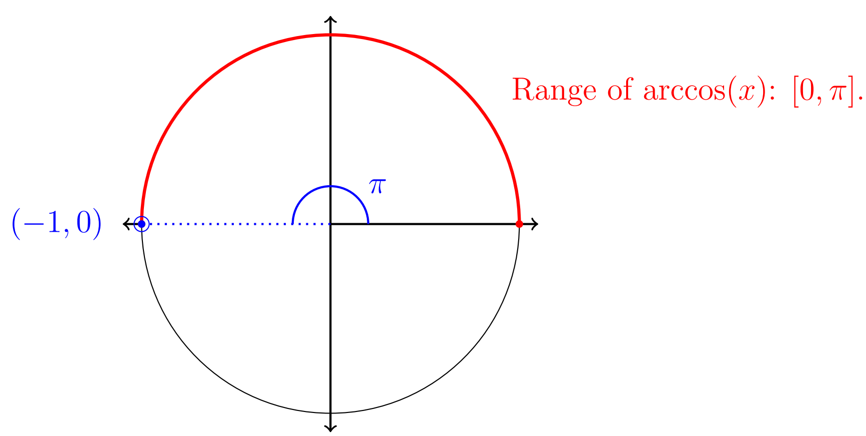


Figure 6: Calculating $\arccos(\cos(-\pi))$.

9. $\arctan(\tan(5\pi/4)) = \pi/4$, as shown in the figure below.

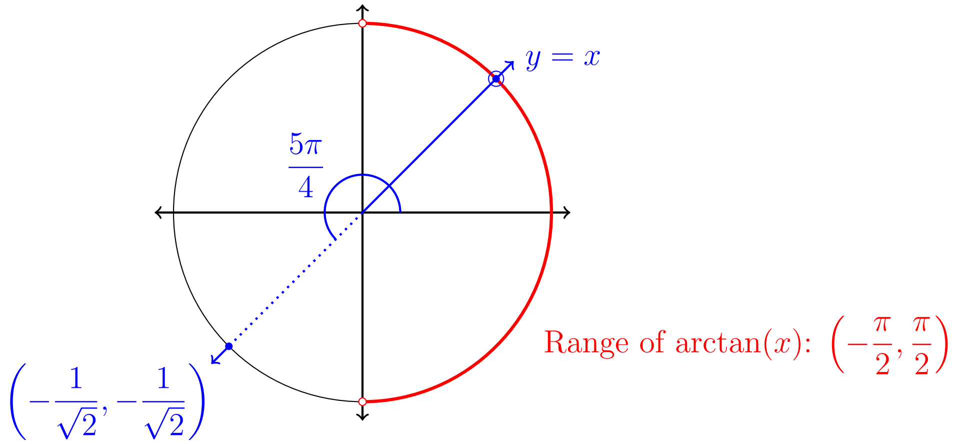


Figure 7: Calculating $\arctan(\tan(5\pi/4))$.

10. $\arcsin(\sin(4\pi/3)) = -\pi/3$, as shown in the figure below.

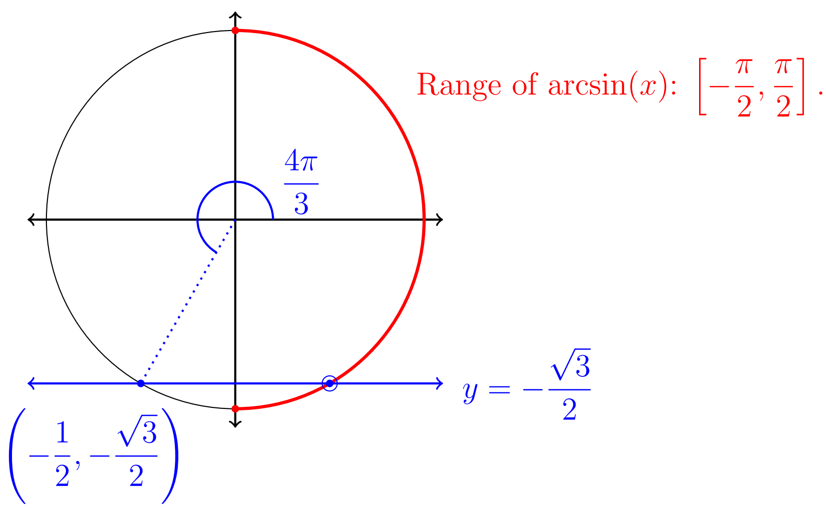


Figure 8: Calculating $\arcsin(\sin(4\pi/3))$.