Tangents to Curves

Up to this point, we used calculus and derivatives to look at properties of graphs of functions. But what about curves which are *not* graphs of functions? That is, curves that do not pass the vertical line test?

A simple example is a circle, such as $x^2 + y^2 = 9$. There are two ways we can approach this. First, we can solve for y and look at the top and bottom halves of the circle separately.

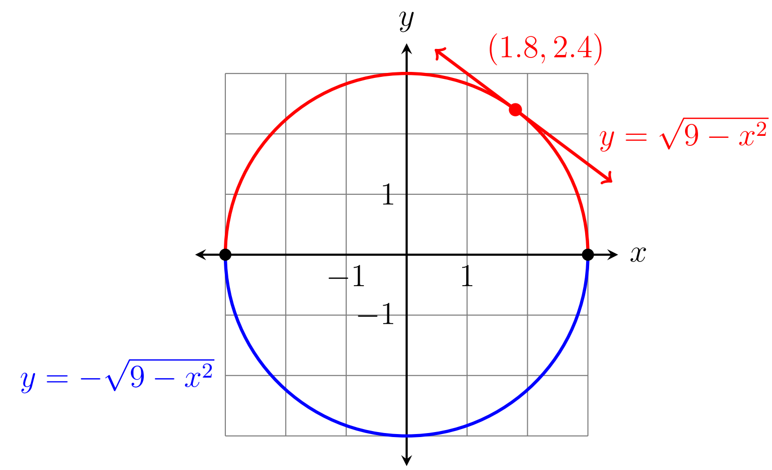

Figure 1: Describing a circle using two functions.

Each half is the graph of a function, as shown in Figure 1. Let's find an equation for the tangent line at $(1.8, 2.4)$. We'll use the Chain Rule with $h(x) = \sqrt{9 - x^2}$, using $f(x) = \sqrt{x}$ and $g(x) = 9 - x^2$, and so $f'(x) = \dfrac{1}{2\sqrt{x}}$ and $g'(x) = -2x$.

$$
\begin{aligned}
h(x) &= \sqrt{9 - x^2} \\
h'(x) &= f'(g(x))g'(x) \\
&= \frac{1}{2\sqrt{g(x)}} \cdot (-2x) \\
&= -\frac{2x}{2\sqrt{9 - x^2}} \\
&= -\frac{x}{\sqrt{9 - x^2}}
\end{aligned}
$$

Plugging in $x = 1.8$, we get $h'(1.8) = -0.75$. To find an equation for the tangent line, we use the point-slope formula.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2.4 &= -0.75(x - 1.8) \\ y - 2.4 &= -0.75x + 1.35 \\ y &= -0.75x + 3.75 \\ y &= -\frac{1}{4}x + \frac{15}{4} \end{aligned}$$

What made this problem a lot of work was having to work with the square root. It turns out there is an easier way using a method called **implicit differentiation.** If we start with $y = f(x)$ and take the derivative, we get $\dfrac{dy}{dx} = f'(x)$. In other words, differentiate both sides:

$$\begin{aligned} y &= f(x) \\ \frac{d}{dx}y &= \frac{d}{dx}f(x) \\ \frac{dy}{dx} &= f'(x) \end{aligned}$$

Now if we don't have the graph of a function, we cannot use $f'(x)$. So when we perform implicit differentiation, we use the notation $\dfrac{dy}{dx}$. Let's begin with our original equation and differentiate both sides:

$$\begin{aligned} x^2 + y^2 &= 9 \\ \frac{d}{dx}x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}9 \\ 2x + \frac{d}{dx}y^2 &= 0 \end{aligned}$$

Two of the terms are easy, but what do we do with $\dfrac{d}{dx}y^2$? Here, we need the Chain Rule. Let's call $p(x) = y^2$, so $f(x) = x^2$ and $g(x) = y$. Then $f'(x) = 2x$ and $g'(x) = \dfrac{dy}{dx}$. So

$$\begin{aligned} p(x) &= y^2 \\ p'(x) &= \frac{d}{dx}y^2 \\ &= f'(g(x))g'(x) \\ &= f'(y)\frac{dy}{dx} \\ &= 2y\frac{dy}{dx} \end{aligned}$$

Let's look at this for a moment:

$$\frac{d}{dx}y^2 = 2y\frac{dy}{dx}.$$

It looks like we differentiate y^2 where y is the variable, and then multiply by $\dfrac{dy}{dx}$. Looking back at the Chain Rule, that's *exactly* what we did. So

$$\frac{d}{dx}\sin(y) = \cos(y)\frac{dy}{dx}, \quad \frac{d}{dx}e^y = e^y\frac{dy}{dx}, \quad \frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx},$$

and so on. Once we observe this pattern, we can skip the Chain Rule each time. But it's important to see where the pattern comes from.

Going back, let's substitute $\dfrac{d}{dx}y^2 = 2y\dfrac{dy}{dx}$ back in and solve for $\dfrac{dy}{dx}$.

$$\begin{aligned} 2x + \frac{d}{dx}y^2 &= 0 \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= -\frac{x}{y} \end{aligned}$$

This is much simpler algebraically. Also notice that it is the same result, since because $y = \sqrt{9 - x^2}$, then

$$-\frac{x}{y} = -\frac{x}{\sqrt{9 - x^2}}.$$

Also, the slope at $(1.8, 2.4)$ is easier to find:

$$-\frac{x}{y} = -\frac{1.8}{2.4} = -\frac{3}{4}.$$

So often, implicit differentiation can be much easier to use. Sometimes, it is *impossible* to solve for y, as we will see later, and then there is no other viable option.

Example 1

Consider the ellipse defined by $x^2 + xy + y^2 = 12$, which you can see graphed at desmos.com. Give equations for the horizontal and vertical tangents to this curve.

Implicit differentiation is very helpful here, since otherwise, we would need to use the quadratic formula to solve for y, which would make the algebra really messy. So let's first find $\dfrac{dy}{dx}$. We will use the fact that $\dfrac{d}{dx}y^2 = 2y\dfrac{dy}{dx}$ from the last problem.

$$\begin{aligned} x^2 + xy + y^2 &= 12 \\ \frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^2 &= \frac{d}{dx}12 \\ 2x + \frac{d}{dx}xy + 2y\frac{dy}{dx} &= 0 \end{aligned}$$

How do we handle $\dfrac{d}{dx}xy$? Here, we need the product rule. Since we're using the notation $\dfrac{dy}{dx}$, let's rewrite the Product Rule first. Suppose $h(x) = f(x)g(x)$.

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ \frac{d}{dx}h(x) &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \end{aligned}$$

Now put $h(x) = xy$, with $f(x) = x$ and $g(x) = y$, so that $f'(x) = 1$ and $g'(x) = \dfrac{dy}{dx}$. Then

$$\begin{aligned} \frac{d}{dx}h(x) &= \frac{d}{dx}xy \\ &= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \\ &= x \cdot \frac{dy}{dx} + y \cdot 1 \\ &= x\frac{dy}{dx} + y \end{aligned}$$

Now let's substitute back in and solve for $\dfrac{dy}{dx}$.

$$\begin{aligned} 2x + \frac{d}{dx}xy + 2y\frac{dy}{dx} &= 0 \\ 2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} &= 0 \\ x\frac{dy}{dx} + 2y\frac{dy}{dx} &= -(2x + y) \\ (x + 2y)\frac{dy}{dx} &= -(2x + y) \\ \frac{dy}{dx} &= \frac{-(2x + y)}{x + 2y} \end{aligned}$$

Let's see how we use this to find horizontal and vertical tangents. For a horizontal tangent, we need $\dfrac{dy}{dx} = 0$, and so the numerator of $\dfrac{dy}{dx}$ must be 0. Then

$$\begin{aligned} -(2x + y) &= 0 \\ 2x + y &= 0 \\ y &= -2x \end{aligned}$$

What does this mean? In desmos, make sure that the ellipse and the equation $y = -2x$ are selected. You can see immediately that they intersect at the points where there are horizontal tangents. So these points must satisfy the equations:

$$\begin{aligned} x^2 + xy + y^2 &= 12 \\ y &= -2x \end{aligned}$$

So to find the points, we can substitute in and solve.

$$\begin{aligned} x^2 + xy + y^2 &= 12 \\ x^2 + x(-2x) + (-2x)^2 &= 12 \\ x^2 - 2x^2 + 4x^2 &= 12 \\ 3x^2 &= 12 \\ x^2 &= 4 \\ x &= -2, +2 \end{aligned}$$

And since $y = -2x$, the two points where there are horizontal tangents are $(-2, 4)$ and $(2, -4)$, as we can see from looking at the graph.

This looks like a *lot* of algebra – and it is. But the important point is that we're not really using any *new* formulas, just applying old formulas in a new situation.

What about the vertical tangents? A vertical tangent has undefined slope. This means the denominator of $\dfrac{dy}{dx}$ must be 0.

$$\begin{aligned} x + 2y &= 0 \\ x &= -2y \\ y &= -\frac{1}{2}x \end{aligned}$$

In desmos, make sure that the ellipse and the equation $y = -\dfrac{1}{2}x$ are selected. You can see right away that they intersect at the points where there are vertical tangents. So these points must satisfy the equations:

$$\begin{aligned} x^2 + xy + y^2 &= 12 \\ x &= -2y \end{aligned}$$

Notice that we use the equation in the form $x = -2y$. This is because to graph, we solve $y = -\dfrac{1}{2}x$. But to substitute, it's easier not to have to worry about fractions.

$$
\begin{aligned}
x^2 + xy + y^2 &= 12 \\
(-2y)^2 + (-2y)y + y^2 &= 12 \\
4y^2 - 2y^2 + y^2 &= 12 \\
3y^2 &= 12 \\
y^2 &= 4 \\
y &= -2, +2
\end{aligned}
$$

Since $x = -2y$, the two points where there are vertical tangents are $(4, -2)$ and $(-4, 2)$. We can see this from looking at the graph.

Example 2 Show that the curve described by the equation $e^{xy} = x + y$ has a horizontal tangent at $(0, 1)$ and a vertical tangent $(1, 0)$.

A graph of this equation is in the desmos notebook. You can see the plausibility of horizontal and vertical tangents at the points $(0, 1)$ and $(1, 0)$, respectively.

We begin by taking derivatives on both sides, as with the other example.

$$\begin{aligned} e^{xy} &= x + y \\ \frac{d}{dx}e^{xy} &= \frac{d}{dx}x + \frac{d}{dx}y \\ \frac{d}{dx}e^{xy} &= 1 + \frac{dy}{dx} \end{aligned}$$

Now we need to deal with the $\frac{d}{dx}e^{xy}$ term. First, we need to use Chain Rule. We use $h(x) = e^{xy}$, so that $f(x) = e^x$ and $g(x) = xy$. Then $f'(x) = e^x$, and from the last example, we have $\frac{d}{dx}g(x) = \frac{d}{dx}xy = x\frac{dy}{dx} + y$. Then

$$\begin{aligned} \frac{d}{dx}e^{xy} &= f'(g(x))g'(x) \\ &= e^{g(x)} \cdot \left(x\frac{dy}{dx} + y \right) \\ &= e^{xy} \left(x\frac{dy}{dx} + y \right) \end{aligned}$$

Now let's substitute back in and solve for $\frac{dy}{dx}$.

$$\begin{aligned} \frac{d}{dx}e^{xy} &= 1 + \frac{dy}{dx} \\ e^{xy}\left(x\frac{dy}{dx} + y \right) &= 1 + \frac{dy}{dx} \\ xe^{xy}\frac{dy}{dx} + ye^{xy} &= 1 + \frac{dy}{dx} \\ xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} &= 1 - ye^{xy} \\ (xe^{xy} - 1)\frac{dy}{dx} &= 1 - ye^{xy} \\ \frac{dy}{dx} &= \frac{1 - ye^{xy}}{xe^{xy} - 1} \end{aligned}$$

This seems like a lot of work, but there is no way to solve this equation for y. So here, implicit differentiation *must* be used.

Now let's see how we use this to answer the question of tangents. Let's find $\frac{dy}{dx}$ at the point $(0, 1)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 - ye^{xy}}{xe^{xy} - 1} \\ &= \frac{1 - 1 \cdot e^{0 \cdot 1}}{0 \cdot e^{0 \cdot 1} - 1} \\ &= \frac{1 - 1}{0 - 1} \\ &= 0.\end{aligned}$$

This means that there is a horizontal tangent at $(0, 1)$. Let's look at what happens at the point $(1, 0)$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 - ye^{xy}}{xe^{xy} - 1} \\ &= \frac{1 - 0 \cdot e^{1 \cdot 0}}{1 \cdot e^{1 \cdot 0} - 1} \\ &= \frac{1 - 0}{1 - 1} \\ &= \frac{1}{0}.\end{aligned}$$

24 This slope is undefined, which means there is a vertical tangent at $(1, 0)$.

Homework

NOTE: Please do not jump straight to the Solutions if you are stuck. Go back and study the Examples, since each problem is similar to one of the Examples. You *will* have a problem on the next Exam of the form "Show that $\frac{dy}{dx} = \ldots$" This means if you don't get the answer right the first time, you'll have to go back and recheck your computations. Practice doing this with the Homework so you can do well on that question.

1. Consider the ellipse $4x^2 + y^2 = 8$. Graph this on desmos.
   (a) Show that $\dfrac{dy}{dx} = -\dfrac{4x}{y}$.
   (b) Find an equation of the tangent line at the point $(-1, 2)$. Check you answer by graphing it on desmos as well.
2. Graph the hyperbola $2x^2 - xy - y^2 = 9$ on desmos.
   (a) Show that
   $$\frac{dy}{dx} = \frac{4x - y}{x + 2y}.$$
   (b) Use this to find where there are vertical tangents to the hyperbola. Verify this on desmos.
3. Graph $e^{x^2y} = y$ on desmos. It looks like there is a horizontal tangent at the point $(0, 1)$.
   (a) Verify that the point $(0, 1)$ is on the curve.
   (b) Show that $\dfrac{dy}{dx} = \dfrac{2xye^{x^2y}}{1 - x^2e^{x^2y}}$.
   (c) Verify that there is a horizontal tangent at the point $(0, 1)$.

Solutions

1. (a) First, find $\dfrac{dy}{dx}$.

$$\begin{aligned}
4x^2 + y^2 &= 8 \\
\frac{d}{dx}4x^2 + \frac{d}{dx}y^2 &= \frac{d}{dx}8 \\
8x + 2y\frac{dy}{dx} &= 0 \\
2y\frac{dy}{dx} &= -8x \\
\frac{dy}{dx} &= \frac{-8x}{2y} \\
&= -\frac{4x}{y}
\end{aligned}$$

(b) At the point $(-1, 2)$,

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{4x}{y} \\
&= -\frac{4(-1)}{2} \\
&= 2.
\end{aligned}$$

Using the point-slope equation of a line:

$$\begin{aligned}
y - y_1 &= m(x - x_1) \\
y - 2 &= 2(x - (-1)) \\
y - 2 &= 2x + 2 \\
y &= 2x + 4
\end{aligned}$$

2. (a) First, find $\dfrac{dy}{dx}$.

$$\begin{aligned}
2x^2 - xy - y^2 &= 9 \\
\frac{d}{dx}2x^2 - \frac{d}{dx}xy - \frac{d}{dx}y^2 &= \frac{d}{dx}9 \\
4x - x\frac{dy}{dx} - y - 2y\frac{dy}{dx} &= 0 \\
-x\frac{dy}{dx} - 2y\frac{dy}{dx} &= -(4x - y) \\
-(x + 2y)\frac{dy}{dx} &= -(4x - y) \\
\frac{dy}{dx} &= \frac{-(4x - y)}{-(x + 2y)} \\
&= \frac{4x - y}{x + 2y}
\end{aligned}$$

(b) There are vertical tangents when the denominator of $\dfrac{dy}{dx}$ is 0.

$$\begin{aligned}
x + 2y &= 0 \\
x &= -2y
\end{aligned}$$

It is easier to solve for x here to avoid fractions. Substitute this back in and solve for y.

$$\begin{aligned}
2x^2 - xy - y^2 &= 9 \\
2(-2y)^2 - (-2y)y - y^2 &= 9 \\
8y^2 + 2y^2 - y^2 &= 9 \\
9y^2 &= 9 \\
y &= -1, +1
\end{aligned}$$

We can find x since we know that $x = -2y$. So there are vertical tangents at the points $(2, -1)$ and $(-2, 1)$. This can be visually verified by looking at the graph.

3. (a) Substituting in:

$$\begin{aligned} e^{x^2y} &= y \\ e^{0^2 \cdot 1} &= 1 \\ e^0 &= 1 \end{aligned}$$

(b) Now find $\dfrac{dy}{dx}$.

$$\begin{aligned} e^{x^2y} &= y \\ \frac{d}{dx}e^{x^2y} &= \frac{d}{dx}y \\ e^{x^2y}\frac{d}{dx}x^2y &= \frac{dy}{dx} \\ e^{x^2y}\left(x^2\frac{dy}{dx} + 2xy\right) &= \frac{dy}{dx} \\ x^2e^{x^2y}\frac{dy}{dx} + 2xye^{x^2y} &= \frac{dy}{dx} \\ 2xye^{x^2y} &= \frac{dy}{dx} - x^2e^{x^2y}\frac{dy}{dx} \\ 2xye^{x^2y} &= \frac{dy}{dx}\left(1 - x^2e^{x^2y}\right) \\ \frac{dy}{dx} &= \frac{2xye^{x^2y}}{1 - x^2e^{x^2y}} \end{aligned}$$

(c) Substitute the point $(0, 1)$ into $\dfrac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2xye^{x^2y}}{1 - x^2e^{x^2y}} \\ &= \frac{2 \cdot 0 \cdot 1e^{0^2 \cdot 1}}{1 - 0^2e^{0^2 \cdot 1}} \\ &= \frac{0}{1 - 0} \\ &= 0. \end{aligned}$$

Since the slope is 0, there is a horizontal tangent at $(0, 1)$. Note: If we were not given a graph, we would have to look at the second derivative, but we will not be doing that. For a problem like this, you would be given a graph.