Asymptotes and Limits at Infinity, III

We saw that $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} = 0$, which meant that e^x grows much faster than x^2 as $x \to \infty$. There is nothing special about the exponent of "2," and in fact, we have:

$$\boxed{\lim_{x\to\infty} \frac{x^n}{e^x} = 0, \quad n > 0.}$$

It is very important to realize that this faster growth is as $x \to \infty$. To see why, go to desmos.com. We will look at the horizontal asymptotes of $f(x) = \dfrac{x}{x+e^x}$.

Looking as $x \to \infty$, we see that this limit is of the form $\dfrac{\infty}{\infty}$, and so we may apply L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{x}{x+e^x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1}{1+e^x} = 0.$$

We can observe this visually on the graph. It also makes sense, since e^x grows faster than x, and the e^x is in the denominator.

Looking at $x \to -\infty$, we see that this limit is of the form $\dfrac{-\infty}{-\infty}$. This is because $\lim\limits_{x\to-\infty} e^x = 0$, meaning the x is the dominant term in the denominator. So for x far to the left,

$$\frac{x}{x+e^x} \approx \frac{x}{x} = 1.$$

We can also see this using L'Hôpital's Rule:

$$\lim_{x\to-\infty} \frac{x}{x+e^x} \overset{\text{LR}}{=} \lim_{x\to-\infty} \frac{1}{1+e^x} = \frac{1}{1+0} = 1.$$

Thus, as we look to the left, we see a horizontal asymptote at $y = 1$. We summarize these observations.

Suppose $n > 0$. Then:

1. as $x \to \infty$, e^x dominates x^n, and
2. as $x \to -\infty$, x^n dominates e^x (if x^n is well-defined).

By well-defined, we mean that x^n exists. For example, when $x < 0$, $x^{1/2} = \sqrt{x}$ is not defined, but $x^{1/3} = \sqrt[3]{x}$ is defined.

We could undertake a similar investigation with $\ln x$, but suffice it to say that

$$\boxed{\lim_{x\to\infty} \frac{\ln x}{x^n} = 0, \quad n > 0.}$$

Exponentials and Logarithms

We've compared powers of x to e^x and $\ln x$, but what about other bases? Remember, $\ln x = \log_e x$. For example, what about

$$\lim_{x\to\infty} \frac{x^2}{2^x}?$$

If we want to use L'Hôpital's Rule, we need to be able to take the derivative of $h(x) = 2^x$. To do this, we observe that since e^x and $\ln x$ are inverse functions, then $2 = e^{\ln 2}$. This means that

$$2^x = \left(e^{\ln 2}\right)^x = e^{(\ln 2)x}.$$

So we can write $h(x)$ as $f(g(x))$, where $f(x) = e^x$ and $g(x) = (\ln 2)x$. Then $f'(x) = e^x$ and $g'(x) = \ln 2$. Using the chain rule, we have

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= e^{g(x)} \ln 2 \\ &= e^{(\ln 2)x} \ln 2 \\ &= 2^x \ln 2. \end{aligned}$$

Now we can use this in L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{x^2}{2^x} \overset{\text{LR}}{=} \frac{2x}{2^x \ln 2} \overset{\text{LR}}{=} \frac{2}{2^x(\ln 2)(\ln 2)} = 0.$$

We can see this by looking at the graphs in desmos. So as long as the base is larger than 1 (or else the exponential function is *decreasing*), these quotients behave similarly. Since the same calculations work for any base:

> Suppose $b > 1$ and $n > 0$. Then:
>
> 1. $\dfrac{d}{dx} b^x = b^x \ln b$,
> 2. $\displaystyle\lim_{x\to\infty} \frac{x^n}{b^x} = 0$,
> 3. as $x \to \infty$, b^x dominates x^n, and
> 4. as $x \to -\infty$, x^n dominates b^x (if x^n is well-defined).

Logarithms can also occur in other bases, as in

$$\lim_{x\to\infty} \frac{\log_2 x}{\sqrt{x}}.$$

We can tackle limits like these by remembering the change of base formula for logarithms:

> Suppose b, $c > 0$. Then for *any* other base $a > 1$,
> $$\log_b c = \frac{\log_a c}{\log_a b}.$$

Why this helps is that we know all about the base e. So we can use $a = e$ in the change of base formula, giving

> Suppose b, $c > 0$. Then
> $$\log_b c = \frac{\ln c}{\ln b}.$$

Let's use this to take the derivative of $p(x) = \log_2 x$. Remember that $\ln 2$ is just a constant.

$$\begin{aligned} p(x) &= \log_2 x \\ &= \frac{\ln x}{\ln 2} \\ p'(x) &= \frac{1}{x \ln 2} \end{aligned}$$

Now we can use this to evaluate our limit. Note that it is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\log_2 x}{\sqrt{x}} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{x\ln 2}}{\dfrac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{x \ln 2} \cdot \frac{2\sqrt{x}}{1} = \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln 2} = 0.$$

The same calculations work for any base, so:

> Suppose $b > 1$ and $n > 0$. Then:
>
> 1. $\dfrac{d}{dx} \log_b x = \dfrac{1}{x \ln b},$
> 2. $\displaystyle\lim_{x\to\infty} \frac{\log_b x}{x^n} = 0$, and
> 3. as $x \to \infty$, x^n dominates $\log_b x$.

L'Hôpital's Rule Disguised

Now we'll look at some example of limits which, although *not* of the form that allows us to apply L'Hôpital's Rule, they can be rewritten so L'Hôpital's Rule can be applied. The idea is similar to when we rewrote $\dfrac{1}{x^4}$ as x^{-4} so we could use the Power Rule instead of the Quotient Rule. Best to start with an example.

Find $\lim\limits_{x\to\infty}(x+1)e^{-3x}$. The difficulty with limits like these is that one part blows up, and the other goes to 0. Here, $\lim\limits_{x\to\infty}(x+1)$ DNE $(+\infty)$ and $\lim\limits_{x\to\infty} e^{-3x} = 0$. We need to see what one "wins." We call this type of limit "$0\cdot\infty$" or "$\infty\cdot 0$."

But isn't 0 times anything equal to 0? Yes, if that anything is a *number.* But ∞ is *not* a number, but represents numbers getting larger than larger. Below are three limits of the form $\infty\cdot 0$.

$$\begin{aligned}
\lim_{x\to\infty} x\cdot\frac{1}{x^2} &= \lim_{x\to\infty}\frac{1}{x} = 0,\\
\lim_{x\to\infty} x\cdot\frac{1}{x} &= \lim_{x\to\infty} 1 = 1,\\
\lim_{x\to\infty} x^2\cdot\frac{1}{x} &= \lim_{x\to\infty} x \text{ DNE } (+\infty).
\end{aligned}$$

So a limit of the form $\infty\cdot 0$ can be 0, a nonzero number, or might not even exist. So you can't automatically say it's 0.

So what can we do? Remember, a negative exponent lets us move an expression to the denominator. So

$$\lim_{x\to\infty}(x+1)e^{-3x} = \lim_{x\to\infty}\frac{x+1}{e^{3x}}.$$

Now it is of the form $\dfrac{\infty}{\infty}$, so L'Hôpital's Rule can be applied.

$$\lim_{x\to\infty}\frac{x+1}{e^{3x}} \overset{\text{LR}}{=} \lim_{x\to\infty}\frac{1}{3e^{3x}} = 0.$$

Let's try another: $\lim\limits_{x\to 0^+} x\ln x$. As you can see in the desmos notebook, this limit should be 0, but we'll use calculus to show it. This limit is of the form $0\cdot\infty$. Note that 0^+ is needed since $\ln x$ is not defined for $x\le 0$. There's no negative exponent here, so we have two options:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{x}{1/\ln x} = \lim_{x\to 0^+}\frac{\ln x}{1/x}.$$

The last two limits are of the form $\dfrac{\infty}{\infty}$. Which one should we try? Let's take them one at a time. For the first, we'll need to take the derivative of $(\ln x)^{-1}$, so let's use the Chain

Rule first before applying L'Hôpital's Rule. We write $h(x) = (\ln x)^{-1}$ as $f(g(x))$, where $f(x) = \dfrac{1}{x} = x^{-1}$ and $g(x) = \ln x$. So $f'(x) = -x^{-2}$ and $g'(x) = \dfrac{1}{x}$.

$$\begin{aligned}
h(x) &= f(g(x)) \\
h'(x) &= f'(g(x))g'(x) \\
&= -(g(x))^{-2} \cdot \frac{1}{x} \\
&= -(\ln x)^{-2} \cdot \frac{1}{x} \\
&= -\frac{1}{x(\ln x)^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0^+} \frac{x}{1/\ln x} &\overset{\text{LR}}{=} \frac{1}{-\dfrac{1}{x(\ln x)^2}} \\
&= \lim_{x\to 0+} -x(\ln x)^2
\end{aligned}$$

The problem? This limit is *still* of the form $0 \cdot \infty$. And instead of a $\ln x$, we have a $(\ln x)^2$, which seems to make the problem worse. So let's try the other way.

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} \overset{\text{LR}}{=} \lim_{x\to 0^+} \frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x} \cdot \frac{-x^2}{1} = \lim_{x\to 0^+} (-x) = 0.$$

You'll notice that the first way only made the problem harder, but the second way wasn't too difficult. So how do you decide? The rule of thumb is that if you have a limit like this involving $\ln x$, just leave the $\ln x$ where it is, and move the other term.

Homework

1. Find the derivatives of the following functions.
   (a) $h(x) = 5^x$
   (b) $h(x) = \log_3 x$
   (c) $h(x) = \log_2(x^2 + 1)$
   (d) $h(x) = 4^{3x+1}$
   (e) $h(x) = \log_5(x^2 5^x)$
2. Find the following limits.
   (a) $\lim_{x\to\infty} e^{-3x} \ln x$
   (b) $\lim_{x\to\infty} \frac{4^x}{3^x}$
   (c) $\lim_{x\to-\infty} \frac{4^x}{3^x}$
   (d) $\lim_{x\to-\infty} e^{2x} x^2$
   (e) $\lim_{x\to 0^+} x^2 \ln x$

Solutions

1. (a) $h'(x) = 5^x \ln 5$

   (b) $h'(x) = \dfrac{1}{x \ln 3}$

   (c) Use the Chain Rule with $f(x) = \log_2 x$ and $g(x) = x^2 + 1$. Then $f'(x) = \dfrac{1}{x \ln 2}$ and $g'(x) = 2x$. Therefore,

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= \frac{1}{g(x)\ln 2} \cdot 2x \\ &= \frac{2x}{(x^2+1)\ln 2} \end{aligned}$$

   (d) Use the Chain Rule with $f(x) = 4^x$ and $g(x) = 3x + 1$. Then $f'(x) = 4^x \ln 4$ and $g'(x) = 3$.

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(x) &= f'(g(x))g'(x) \\ &= 4^{g(x)}(\ln 4) \cdot 3 \\ &= 3 \cdot 4^{3x+1} \ln 4 \end{aligned}$$

   (e) Use rules of logarithms to simplify first.

$$f(x) = \log_5(x^2 5^x) = \log_5(x^2) + \log_5(5^x) = 2\log_5(x) + x.$$

   Then

$$h'(x) = \frac{2}{x \ln 5} + 1.$$

2. (a) This limit is of the form $0 \cdot \infty$. Rewrite and use L'Hôpital's Rule.

$$\lim_{x\to\infty} e^{-3x} \ln x = \lim_{x\to\infty} \frac{\ln x}{e^{3x}} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/x}{3e^{3x}} = \lim_{x\to\infty} \frac{1}{3xe^{3x}} = 0.$$

   (b) This limit is of the form $\dfrac{\infty}{\infty}$. Using L'Hôpital's Rule, we get

$$\lim_{x\to\infty} \frac{4^x}{3^x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{4^x \ln 4}{3^x \ln 3}.$$

   This is still of the form $\dfrac{\infty}{\infty}$, and using L'Hôpital's Rule again will not help. But using rules of exponents,

$$\frac{4^x}{3^x} = \left(\frac{4}{3}\right)^x.$$

So

$$\lim_{x\to\infty}\left(\frac{4}{3}\right)^x \text{ DNE } (+\infty),$$

57 since we are taking a number which is greater than 1 to larger and larger powers.

(c) This limit is of the form $\dfrac{0}{0}$. Using L'Hôpital's Rule, we get

$$\lim_{x\to-\infty}\frac{4^x}{3^x} \overset{\text{LR}}{=} \lim_{x\to-\infty}\frac{4^x\ln 4}{3^x\ln 3}.$$

This is still of the form $\dfrac{0}{0}$, and using L'Hôpital's Rule again will not help. But using rules of exponents,

$$\frac{4^x}{3^x} = \left(\frac{4}{3}\right)^x.$$

So

$$\lim_{x\to-\infty}\left(\frac{4}{3}\right)^x = 0,$$

58 since we are taking a number which is greater than 1 to more negative powers.

(d) This limit is of the form $0\cdot\infty$. So we rewrite as

$$\lim_{x\to-\infty}\frac{x^2}{e^{-2x}} \overset{\text{LR}}{=} \lim_{x\to-\infty}\frac{2x}{-2e^{-2x}} \overset{\text{LR}}{=} \lim_{x\to\infty}\frac{2}{4e^{-2x}} = 0.$$

(e) This limit is of the form $0\cdot\infty$. We move the x^2 term.

$$\lim_{x\to0^+}\frac{\ln x}{x^{-2}} \overset{\text{LR}}{=} \lim_{x\to0^+}\frac{1/x}{-2x^{-3}} = \lim_{x\to0^+}\frac{1/x}{-2/x^3} = \lim_{x\to0^+}\frac{1}{x}\cdot\frac{-x^3}{2} = \lim_{x\to0^+}\frac{-x^2}{2} = 0.$$