Asymptotes and Limits at Infinity, II

For the initial discussion, you will need to visit desmos.com.

We start our discussion by looking at the *growth of functions.* For example, given $f(x) = e^x$ and $g(x) = x^2$, we know that $\lim_{x\to\infty} f(x)$ DNE $(+\infty)$ and $\lim_{x\to\infty} g(x)$ DNE $(+\infty)$. In other words, both functions "blow up" as $x \to \infty$. Is there a way to tell which function blows up faster?

Note: the different lines/functions in the desmos graph are labelled ◯1–◯10. You can select/deselect by clicking on the circles to the left. When you open the graph, you should see ◯2 and ◯3 selected.

The graph of $f(x) = e^x$ is shown. As you move the slider in ◯1, you'll graphs of $g(x) = x$, $g(x) = x^2$, and so on, up to $g(x) = x^{10}$. Once you hit $g(x) = x^3$, it looks like the polynomial is "winning." If you click on the wrench in the upper right and deselect "Lock Viewport," you'll be able to zoom out. If you do, you'll notice it's impossible to tell which function is growing faster.

How can we use limits and calculus to see which function grows faster? One way is to look at the quotient of $f(x)$ and $g(x)$. In other words, we consider $h(x) = \dfrac{x^2}{e^x}$. Make sure that *only* ◯3, ◯4, and ◯5 are selected. Click on the house icon in the upper right to reset the screen.

Now make sure you can tell which graphs are $f(x) = e^x$, $g(x) = x^2$, and $h(x) = \dfrac{x^2}{e^x}$. From looking at the graph, we might guess that

$$\lim_{x\to\infty} h(x) = 0.$$

How can we show this? When we looked at rational functions, we had a definite procedure to follow. When other functions are involved in the quotient, we use a method called **L'Hôpital's Rule**. This method is used for quotients of the form "$\dfrac{\pm\infty}{\pm\infty}$," which means the limits in the numerator and denominator do not exist, but tend towards positive or negative infinity – that is, the limits of the numerator and denominator are either DNE $(+\infty)$ or DNE $(-\infty)$. Here is the result.

> Suppose $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{\pm\infty}{\pm\infty}$. Then
>
> $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
>
> Here, a can be a number or $\pm\infty$.

Let's see how we apply this here. We observed that $\lim_{x\to\infty} x^2$ DNE $(+\infty)$ and $\lim_{x\to\infty} e^x$ DNE $(+\infty)$. So the form is right for using L'Hôpital's Rule. Thus,

$$\lim_{x\to\infty} \frac{x^2}{e^x} \stackrel{\text{LR}}{=} \lim_{x\to\infty} \frac{2x}{e^x},$$

where the notation "$\stackrel{\text{LR}}{=}$" means that the expressions are equal due to L'Hôpital's Rule.

Now you'll notice that $\lim_{x\to\infty} \frac{2x}{e^x}$ is still of the form $\frac{\pm\infty}{\pm\infty}$. So that means we'll have to use L'Hôpital's Rule a second time. This is not unusual when using L'Hôpital's Rule. Therefore,

$$\lim_{x\to\infty} \frac{2x}{e^x} \stackrel{\text{LR}}{=} \lim_{x\to\infty} \frac{2}{e^x}.$$

But now we observe that the numerator is *always* 2, but the denominator tends to $+\infty$. Therefore,

$$\lim_{x\to\infty} \frac{x^2}{e^x} \stackrel{\text{LR}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \stackrel{\text{LR}}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

So this means that $h(x) = \frac{x^2}{e^x}$ does have a horizontal asymptote at $y = 0$. But *only* in the positive direction. From the graph, it appears that $\lim_{x\to-\infty}$ DNE $(+\infty)$. This is because as $x \to -\infty$, the numerator x^2 keeps getting larger and larger, while the denominator e^x goes to 0. Recall that with rational functions, the function always approached the horizontal asymptote from both directions. As we see here, that is not necessarily the case if the quotient is not a rational function.

Essentially, because we take derivatives on the numerator and denominator, L'Hôpital's Rule is saying that to look at the ratio of two expressions tending to $\pm\infty$, we need to compare the *rates* at which these expressions tend to $\pm\infty$.

Now that we have L'Hôpital's Rule, it is important *not* to confuse it with the quotient rule. Remember, to take the derivative of a quotient, you *cannot* just take the derivative of the numerator and the derivative of the denominator. But that's *exactly* what you do when using L'Hôpital's Rule.

> You can use L'Hôpital's Rule to evaluate limits of the form $\frac{\pm\infty}{\pm\infty}$.
> You *cannot* use L'Hôpital's Rule when taking derivatives.

Example 1

We can ask a similar question about $f(x) = \ln x$ and $g(x) = \sqrt{x}$. Make sure only $\bigcirc$6 and $\bigcirc$7 are selected. It looks like $g(x)$ is above $f(x)$ when the home screen icon is clicked. But what about

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}?$$

The logarithm is below the square root so far, but what happens later? If you select $\bigcirc$8 and zoom out, it looks like this ratio could be 0, or perhaps a small number. We can use L'Hôpital's Rule to see just what happens, since for both graphs, $\lim_{x\to\infty} f(x)$ DNE $(+\infty)$ and $\lim_{x\to\infty} g(x)$ DNE $(+\infty)$.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/x}{1/(2\sqrt{x})}.$$

This is starting to look complicated, but when using L'Hôpital's Rule (perhaps more than once), it is important to simplify first. Thus,

$$\frac{1/x}{1/(2\sqrt{x})} = \frac{1}{x} \cdot \frac{2\sqrt{x}}{1} = \frac{2}{\sqrt{x}}.$$

Then we see that $\lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$, since the denominator goes to infinity while the numerator stays at 2. Thus, we conclude that

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0.$$

Let's see geometrically why this makes sense. Select $\bigcirc$3, $\bigcirc$4, $\bigcirc$6, and $\bigcirc$7 only. Notice that e^x and $\ln x$ are inverse functions, as are x^2 (for $x \geq 0$) and $\sqrt{x}$. So if e^x grows *faster* that x^2 as $x \to \infty$, then when you reflect about the line $y = x$ (this is how you transform a graph to get the graph of an inverse function), then $\ln x$ grows *slower* that $\sqrt{x}$.

Example 2

There is a second way we can apply L'Hôpital's Rule. In the previous examples, we did so when the limit was of the form $\frac{\pm\infty}{\pm\infty}$. We can also use L'Hôpital's Rule when the limit is of the form $\frac{0}{0}$. Said using limits, if we want to find $\lim_{x\to 0} \frac{f(x)}{g(x)}$, and if both $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, then L'Hôpital's Rule can be used.

Let's look at

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h}.$$

We used this limit when we used the definition to find the derivative of $\sin(x)$. Let's see how to find the limit using L'Hôpital's Rule.

Since $\cos(0) = 1$, then $\lim_{h\to 0}(\cos(h) - 1) = 0$, and of course $\lim_{h\to 0} h = 0$. This means the limit is of the form $\frac{0}{0}$. So

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} \stackrel{\text{LR}}{=} \lim_{h\to 0} \frac{-\sin(h)}{1} = \lim_{h\to 0}(-\sin(h)).$$

There is no longer a fraction involved, and $\sin(0) = 0$, so this limit is 0; that is

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0.$$

We did say how this could be shown numerically and graphically before, but we can verify it using L'Hôpital's Rule.

Example 3

Now consider a similar limit,

$$\lim_{x\to 0} \frac{\cos(x)-1}{x^2}.$$

Like the previous example, we observe that this limit is of the form $\frac{0}{0}$. Therefore, we may apply L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{\cos(x)-1}{x^2} \overset{\text{LR}}{=} \lim_{x\to 0} \frac{-\sin(x)}{2x}.$$

No need to simplify here, but we need to check what form the limit is. Again, we see that it is of the form $\frac{0}{0}$, so we can apply L'Hôpital's Rule again.

$$\lim_{x\to 0} \frac{-\sin(x)}{2x} \overset{\text{LR}}{=} \lim_{x\to 0} \frac{-\cos(x)}{2}.$$

This time around, nothing is 0, so we can just plug in $x = 0$ here, so

$$\lim_{x\to 0} \frac{\cos(x)-1}{x^2} = -\frac{1}{2}.$$

What does it mean that we've calculated this limit? This means that very near 0, $\frac{\cos(x)-1}{x^2}$ is approximately $-\frac{1}{2}$, which we write as

$$\frac{\cos(x)-1}{x^2} \approx -\frac{1}{2}.$$

Now let's "solve" this for $\cos(x)$ as follows.

$$\begin{aligned} \frac{\cos(x)-1}{x^2} &\approx -\frac{1}{2} \\ \cos(x)-1 &\approx -\frac{1}{2}\cdot x^2 \\ \cos(x) &\approx 1-\frac{x^2}{2}. \end{aligned}$$

Now go back to desmos, and make sure only ◯9 and ◯10 are selected. If you go back to the home screen, you'll see that these functions look very different. But as you zoom in around $x = 0$, you should notice some interesting behavior. The graphs seem to get closer and closer together, and if you zoom in far enough, you can't tell the difference between the two – they look like they're on top of each other. They *only* intersect at $x = 0$, but there's no way to tell this using your computer. It doesn't have good enough resolution for that.

The point is that we discovered that $y = 1-\frac{x^2}{2}$ is a very good approximation to $y = \cos(x)$ by using L'Hôpital's Rule. Finding approximations is such an important application of calculus that one-third of Calculus II is devoted to this topic. We can only scratch the surface here. But as these examples show, using L'Hôpital's Rule is yet another way to study the behavior of graphs.

Summary of L'Hôpital's Rule:

> Suppose $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{\pm\infty}{\pm\infty}$ or of the form $\dfrac{0}{0}$. Then L'Hôpital's Rule tells us that
>
> $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$
>
> Here, a can be a number or $\pm\infty$. L'Hôpital's Rule can *never* be used to evaluate a derivative!

In-Class Practice/Homework

For each of the limits below, decide whether you would be able to apply L'Hôpital's Rule to evaluate. Do *not* evaluate the limits now. For homework, evaluate all the limits where L'Hôpital's Rule may be applied.

1. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^3+1}$
2. $\lim\limits_{x\to-\infty} \dfrac{e^x}{x^3+1}$
3. $\lim\limits_{x\to\infty} \dfrac{x+2}{x^2-4}$
4. $\lim\limits_{x\to 2} \dfrac{x+2}{x^2-4}$
5. $\lim\limits_{x\to\infty} \dfrac{\ln x}{e^{-x}}$
6. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x^2+x}$
7. $\lim\limits_{x\to-\infty} \dfrac{\sin(x)}{x^2+x}$
8. $\lim\limits_{x\to\pi} \dfrac{\sin(x)}{1+\cos(x)}$
9. $\lim\limits_{x\to 0} \dfrac{\sin(x)}{1+\cos(x)}$
10. $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x^2-1}$
11. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x}-1}{x^2-1}$
12. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x}-1}{x^2-1}$
13. $\lim\limits_{x\to\infty} \dfrac{\ln(\ln x))}{x^2+1}$
14. $\lim\limits_{x\to\infty} \dfrac{e^{-x}}{x^2}$
15. $\lim\limits_{x\to-\infty} \dfrac{e^{-x}}{x^2}$

Solutions

1. The limit is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{e^x}{x^3+1} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{e^x}{6x} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{e^x}{6} \text{ DNE } (+\infty).$$

2. The limit is of the form $\dfrac{0}{-\infty}$, so L'Hôpital's Rule does not apply.

3. The limit is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{x+2}{x^2-4} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1}{2x} = 0.$$

4. The limit is of the form $\dfrac{4}{0}$, so L'Hôpital's Rule does not apply.

5. The limit is of the form $\dfrac{\infty}{0}$, so L'Hôpital's Rule does not apply.

6. The limit is of the form $\dfrac{0}{0}$.

$$\lim_{x\to 0} \frac{\sin(x)}{x^2+x} \overset{\text{LR}}{=} \lim_{x\to 0} \frac{\cos(x)}{2x+1} = 1.$$

7. The function $y = \sin(x)$ oscillates continuously between -1 and 1, and so has no limit as $x \to \infty$. So L'Hôpital's Rule cannot be used.

8. The limit is of the form $\dfrac{0}{0}$.

$$\lim_{x\to\pi} \frac{\sin(x)}{1+\cos(x)} \overset{\text{LR}}{=} \lim_{x\to\pi} \frac{\cos(x)}{-\sin(x)} \text{ DNE }.$$

9. The limit is just $\dfrac{0}{2} = 0$. You can just plug in here, and since the denominator tends to 2, L'Hôpital's Rule cannot be applied.

10. The limit is of the form $\dfrac{0}{0}$.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-1} \overset{\text{LR}}{=} \lim_{x\to 1} \frac{1/(2\sqrt{x})}{2x} = \lim_{x\to 1} \frac{1}{4x\sqrt{x}} = \frac{1}{4}.$$

11. The limit is of the form $\dfrac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\sqrt{x}-1}{x^2-1} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{1/(2\sqrt{x})}{2x} = \lim_{x\to\infty} \frac{1}{4x\sqrt{x}} = 0.$$

12. $\sqrt{x}$ is not defined for negative values of x, so this limit is not defined and cannot be evaluated by any method.

13. The limit is of the form $\frac{\infty}{\infty}$. To find the derivative of $\ln(\ln x)$, you'll need the Chain Rule with $f(x) = \ln(x)$ and $g(x) = \ln(x)$.

$$\lim_{x\to\infty} \frac{\ln(\ln x)}{x^2+1} \overset{\text{LR}}{=} \lim_{x\to\infty} \frac{\frac{1}{\ln x} \cdot \frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2 \ln x} = 0.$$

14. The limit is of the form $\frac{0}{\infty}$, so L'Hôpital's Rule cannot be applied.

15. The limit is of the form $\frac{\infty}{\infty}$.

$$\lim_{x\to-\infty} \frac{e^{-x}}{x^2} \overset{\text{LR}}{=} \lim_{x\to-\infty} \frac{-e^{-x}}{2x} \overset{\text{LR}}{=} \lim_{x\to-\infty} \frac{e^{-x}}{2} \text{ DNE } (+\infty).$$