Asymptotes and Limits to Infinity, I

We have described many features of graphs up to the point using limits and calculus. The last features we will describe are horizontal and vertical asymptotes. We'll look at some new notation by examining the graph of $f(x) = \dfrac{1}{x}$.

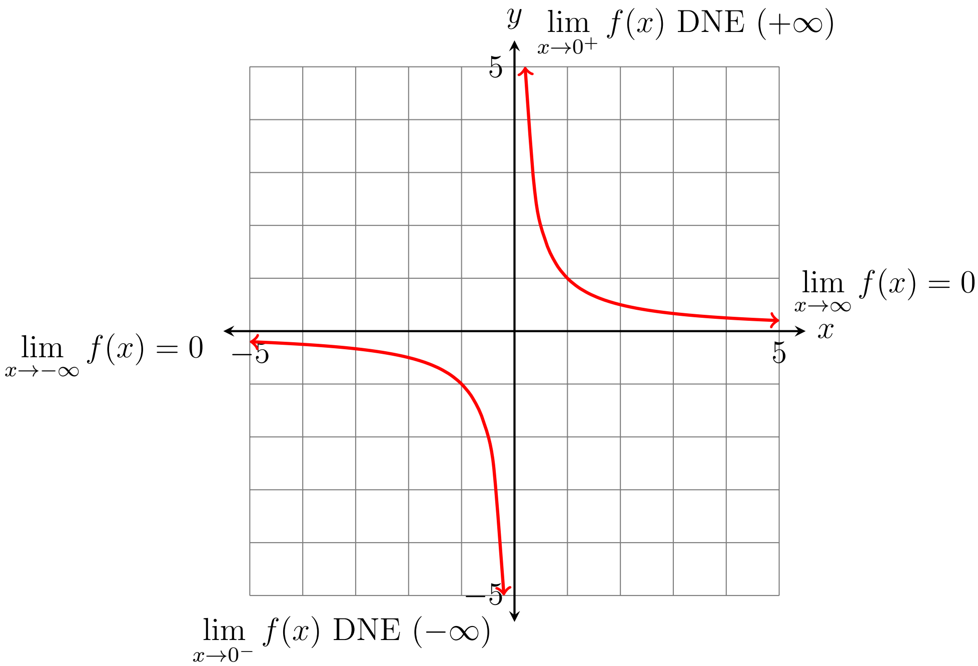

Figure 1: Graph of $f(x) = \dfrac{1}{x}$ with asymptotes described.

Let's see what this new notation means.

1. The $+x$ axis. We read this as "the limit as x goes to infinity of $f(x)$ is 0." Graphically, this means that $f(x)$ is getting closer and closer to 0 as x gets further and further along the x axis. We can also see this using a chart of values, like we did when taking $\displaystyle\lim_{h \to 0} \frac{\sin(h)}{h}$. See Table 3.

   We will look at rules for evaluating such limits later, but the chart is a numerical confirmation that this limit is 0. This means that $y = 0$ is a horizontal asymptote as we look to the right.

x	$1/x$
1	1
10	0.1
100	0.01
1000	0.001
10,000	0.0001

Table 1: Looking at $\lim_{x\to\infty} f(x) = 0$.

2. The $+y$ axis. We read this as "The limit as x goes to 0 from the right does not exist, but it approaches positive infinity," meaning the values of the function keep getting larger and larger without bound.

   Again, we look at a numerical chart to see what's happening. We can see that the

x	$1/x$
1	1
0.1	10
0.01	100
0.001	1000
0.0001	10,000

Table 2: Looking at $\lim_{x\to 0^+} f(x)$ DNE $(+\infty)$.

   values of $f(x)$ will keep getting larger the smaller that x gets. So there is *no* limiting value – the limit does not exist. The notation "$(+\infty)$" indicates that the graph moves *up* along the positive y axis, but never actually touches it.

3. The $-x$ axis. We read this as "The limit as x approaches negative infinity of $f(x)$ is 0." There is no need to make a chart here; it will look very similar to the chart for the $+x$ axis. This means $y = 0$ is also a horizontal asymptote as we look to the left. It is important to observe that here, the curve approaches the asymptote from *below,* but at the $+x$ axis, the curve approaches the asymptote from *above.* The notatation itself does *not* tell you if the curve approaches from above or below, so you need to do additional work to decide which.

4. The $-y$ axis. We read this as "The limit as x approaches 0 from the left does not exist, but it approaches negative infinity." Again, a chart would look very similar to that for the $+y$ axis. As x moves closer and closer to 0 from the left, the graph approaches the

y axis, and keeps going *down* along the y axis, with no lower bound. Thus, there is no lower limit – this is what "DNE $(-\infty)$" means.

The types of functions we'll be looking at are **rational functions,** which are ratios of polynomials; $f(x) = \dfrac{1}{x}$, for example. That is, the numerator and denominator of the function are both polynomials. We'll state how to find asymptotes of such functions, and then look at several examples. Remember that the **degree** of a polynomial is the highest power of x occurring in a polynomial, and that the degree of a constant polynomial (like $f(x) = 5$) is 0, since $5 = 5 \cdot x^0$. Also, the **leading coefficient** of a polynomial is the coefficient of the term with highest degree. So the leading coefficient of $3x^4 - 2x^x + 5x$ is 3.

> Suppose $f(x) = \dfrac{p(x)}{q(x)}$ is a rational function. Let N be the degree of $p(x)$, and D be the degree of $q(x)$. To find the horizontal asymptotes:
>
> 1. If $N > D$, there are no horizontal asymptotes.
> 2. If $N = D$, then there is a horizontal asymptote, in both directions, at $y = c$, where c is the ratio of the leading coefficients of the numerator and denominator.
> 3. If $N < D$, there is a horizontal asymptote, in both directions, at $y = 0$.
>
> To find the vertical asymptotes, first cancel out common factors of the numerator and denominator, if any. Then there are vertical asymptotes where the denominator is equal to 0.

We'll look at several examples which illustrate all of these possibilities.

Example 1

The graph of $f(x) = \dfrac{1}{x^2}$ is shown in Figure 2.

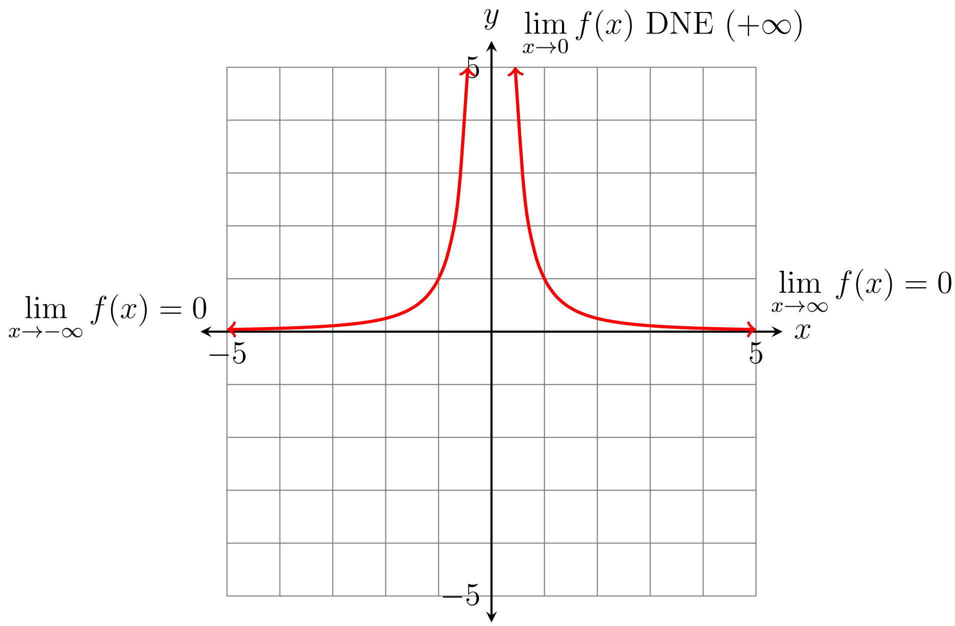

Figure 2: Graph of $f(x) = \dfrac{1}{x^2}$ with asymptotes described.

Let's apply our method here. For the numerator, $N = 0$ since a constant has degree 0, and for the denominator $D = 2$. Since $N < D$, $y = 0$ is a horizontal asymptote.

Now let's look at vertical asymptotes. There are no factors to cancel, so we look at when $x^2 = 0$, which is when $x = 0$. Thus, there is a vertical asymptote at $x = 0$. Because $f(x)$ must always be positive, then

$$\lim_{x \to 0^-} f(x) \text{ DNE } \ (+\infty), \quad \lim_{x \to 0^+} f(x) \text{ DNE } \ (+\infty).$$

Since we have DNE $(+\infty)$ from the left and the right, we can just write

$$\lim_{x \to 0} f(x) \text{ DNE } \ (+\infty).$$

This is similar to how we used the limit notation when looked at one-sided limits earlier. It is important to note that this notation describes the asymptotic behavior at $x = 0$, but it does *not* mean that the limits exist. Remember, DNE means "does not exist."

Example 2

The graph of $f(x) = \dfrac{4x}{x^2+1}$ is shown in Figure 3.

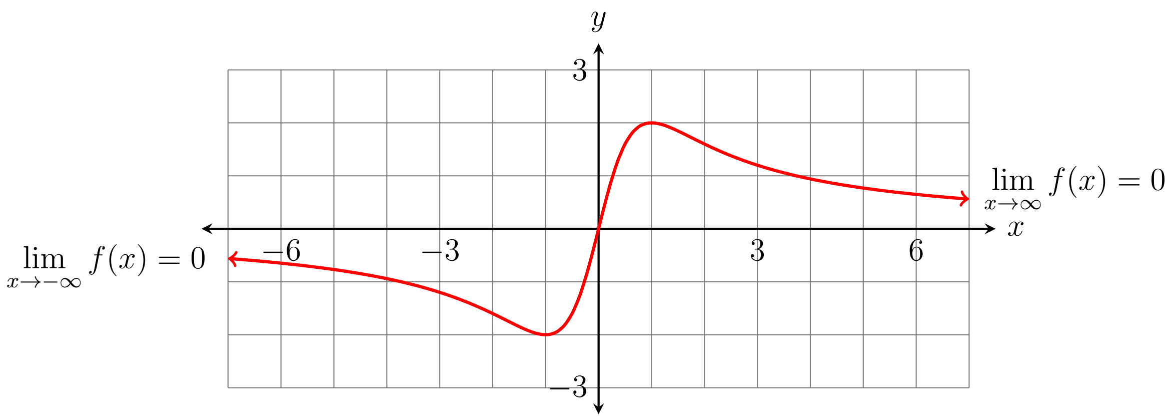

Figure 3: Graph of $f(x) = \dfrac{4x}{x^2+1}$ with asymptotes described.

Let's apply our method here. For the numerator, $N = 1$, and for the denominator $D = 2$. Since $N < D$, $y = 0$ is a horizontal asymptote. Let's take a moment to see why.

As $x \to \infty$, the denominator – having a higher degree – blows up faster than the numerator. This forces the fraction to 0. A few values support this, as seen in the following chart. $f(1) = 2$, but other values are approximate.

x	$f(x)$
1	2
10	0.4
100	0.04
1000	0.004
10,000	0.0004

Table 3: Looking at $\lim_{x\to\infty} f(x) = 0$.

How do we know the graph approaches $y = 0$ from above here? Note that for large, positive x,

$$f(x) = \frac{+}{+} = + > 0,$$

and so we approach from above.

A chart as $x \to -\infty$ looks similar. The graph approaches $y = 0$ from below since for negative x,

$$f(x) = \frac{-}{+} = - < 0,$$

which means we approach from below. These calculations are necessary if you do not have a graph of the function.

Since the denominator can never be 0 ($x^2 + 1$ is always positive), there are no vertical asymptotes.

Example 3

The graph of $f(x) = \dfrac{3x^2 + x}{4x^2 - 4}$ is shown in Figure 4. There is a lot going on here, so let's go one step at a time.

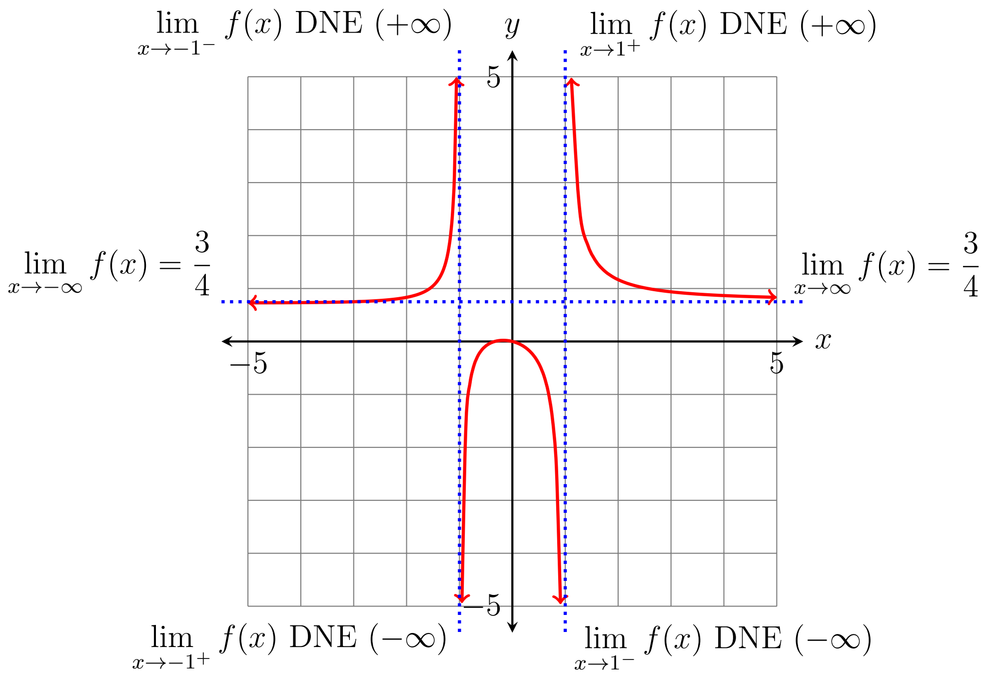

Figure 4: Graph of $f(x) = \dfrac{3x^2 + x}{4x^2 - 4}$ with asymptotes described.

The degree of the numerator is $N = 2$, and the degree of the denominator is $D = 2$. Since $N = D$, we have a horizontal asymptote at the ratio of the leading coefficients, or $y = \dfrac{3}{4}$. Let's see why this makes sense.

The terms with the highest power of x take over as $x \to \infty$. In other words we can say that for large x,

$$f(x) = \frac{3x^2 + x}{4x^2 - 4} \approx \frac{3x^2}{4x^2} = \frac{3}{4}.$$

A brief table of function values (see Table 5) supports this. You can see how the function values approach $\dfrac{3}{4} = 0.75$. Also observe that for values of x far to the left or right, $f(x) = \dfrac{+}{+} = +$, so the graph approaches the horizontal asymptote from above in both directions.

x	$f(x)$
1000	0.750251
10,000	0.750025
100,000	0.750003

Table 4: Looking at $\lim_{x\to\infty} f(x) = \frac{3}{4}$.

Now let's find the vertical asymptotes. First, note that we cannot cancel any factors, since

$$f(x) = \frac{3x^2 + x}{4x^2 - 4} = \frac{x(3x+1)}{4(x+1)(x-1)}.$$

So now see where the denominator is 0, which is when $x = -1$ or $x = 1$. Thus, the lines $x = -1$ and $x = 1$ are vertical asymptotes.

From the graph, it is easy to see the limits as x approaches 1 from the left and right. But suppose you didn't have a graph? We make a brief table of approximate values with numbers close to 1 on either side. These values make the trend clear. Note that the graph approaches

x	$f(x)$
0.99	-49.3756
0.999	-499.375
1.01	50.6256
1.001	500.625

Table 5: Looking at $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

the asymptote in different directions at $x = 1$, and so we write

$$\lim_{x\to 1} f(x) \text{ DNE.}$$

Contrast this to Example 1, where we were able to write

$$\lim_{x\to 0} f(x) \text{ DNE } (+\infty).$$

We observe similar behavior at $x = -1$, and so skip the details.

Example 4

The graph of $f(x) = \dfrac{2x^2-5}{2x-3}$ is shown in Figure 6.

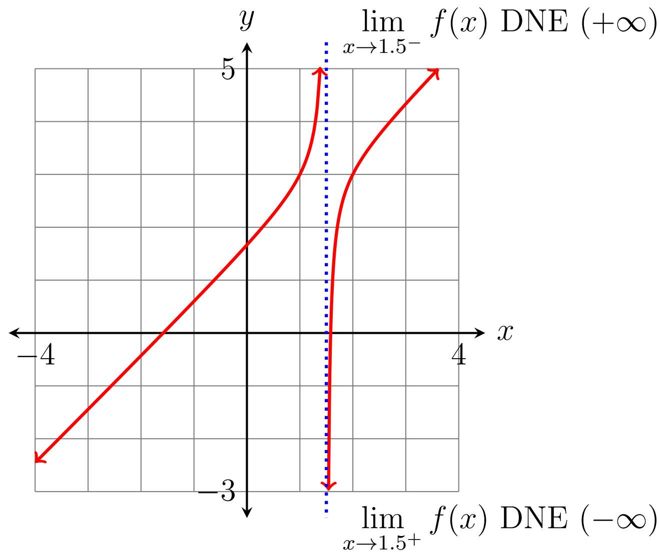

Figure 5: Graph of $f(x) = \dfrac{2x^2-5}{2x-3}$ with asymptotes described.

Let's start with the horizontal asymptotes. The degree of the numerator is $N = 2$ and the degree of the denominator is $D = 1$. Since $N > D$, there are no horizontal asymptotes. This makes sense, since as x moves further out in either direction, we approximate

$$f(x) = \frac{2x^2-5}{2x-3} \approx \frac{2x^2}{2x} = x.$$

Thus, there is no limiting value for $f(x)$ as $x \to \infty$ or $x \to -\infty$.

Now let's look at vertical asymptotes. The only candidate is $x = 3/2$, since this is where the denominator is 0. We do need to check if any factors cancel, though. Now if the $2x - 3$ *did* cancel, we'd need a factor of $2x - 3$ in the numerator. This means that if we plug $x = 3/2$ into the numerator, we'd *also* have to get 0. But

$$2\left(\frac{3}{2}\right)^2 - 5 = -\frac{1}{2} \neq 0,$$

so there is no cancellation. As with Example 3, we can make a brief chart of values to see the behavior of the graph as it approaches the asymptote $x = \dfrac{3}{2}$. We'll skip that here.

Example 5

The graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$ is shown in Figure 6.

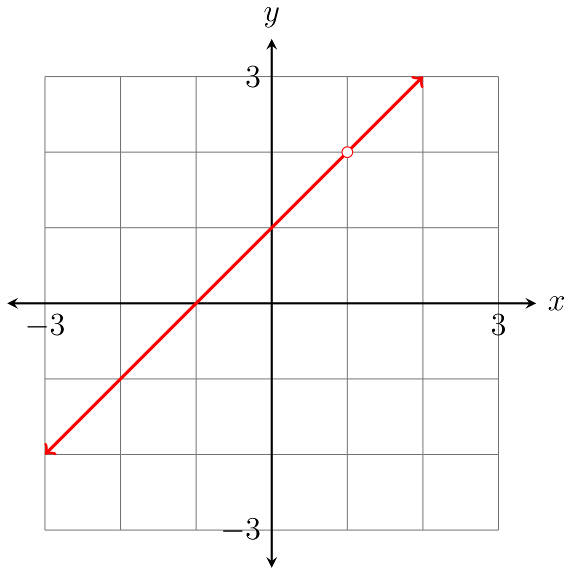

Figure 6: Graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$.

Let's first look for horizontal asymptotes. The degree of the numerator is $N = 2$ and the degree of the denominator is $D = 1$. Since $N > D$, there are no horizontal asymptotes.

At first glance, it looks like there is a vertical asymptote at $x = 1$. But in this case, the numerator factors, and so

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1.$$

So it looks like the graph of $f(x)$ is just a straight line, $y = x + 1$. This is almost correct – but we can't skip over the fact that the original function is not defined at $x = 1$, since we'd get $\dfrac{0}{0}$. So we need to put an open circle at $(1, 2)$. This is because we can use the formula $x + 1$ *only* when $x \neq 1$, since $f(x)$ is undefined at $x = 1$.

Homework

1. For each of the following functions, first graph it is desmos, then find all asymptotes. Describe the behavior at the asymptotes using limit notation, as in the examples in the notes.

    (a) $f(x) = \dfrac{x}{x^2 - 4}$

    (b) $f(x) = \dfrac{x^3 - 1}{x^3 + 1}$

    (c) $f(x) = \dfrac{x^2 - x - 2}{x^2 - 1}$

2. For the function in 1(b), show how you would describe the behavior at the vertical asymptote if you did *not* have a graph.

3. For the function in 1(c), show how you would know if the graph approached the horizontal asymptote from above or below as $x \to -\infty$ and $x \to \infty$ if you did *not* have a graph.

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1. (a) Since $N = 1$ and $D = 2$ and $N < D$, there is a horizontal asymptote at $y = 0$. We write

$$\lim_{x \to -\infty} f(x) = 0, \quad \lim_{x \to \infty} f(x) = 0.$$

No factors cancel out. Since the denominator is 0 when $x = -2, 2$, we have vertical asymptotes at $x = -2$ and $x = 2$. Writing using limits, we have

$$\lim_{x \to -2^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x \to -2^+} f(x) \text{ DNE } (+\infty),$$

and

$$\lim_{x \to 2^-} f(x) \text{ DNE } (-\infty), \quad \lim_{x \to 2^+} f(x) \text{ DNE } (+\infty).$$

(b) Here, $N = 3$ and $D = 3$, and since $N = D$, there is a horizontal asymptote at $y = \frac{1}{1} = 1$. We have

$$\lim_{x \to -\infty} f(x) = 1, \quad \lim_{x \to \infty} f(x) = 1.$$

The denominator is 0 only when $x^3 = -1$, or $x = -1$. Plugging -1 into the numerator, we get $(-1)^3 - 1 = -2 \neq 0$, so we know nothing cancels. In this case, we have

$$\lim_{x \to -1^-} f(x) \text{ DNE } (+\infty), \quad \lim_{x \to -1^+} f(x) \text{ DNE } (-\infty).$$

(c) Here, $N = 2$ and $D = 2$, and since $N = D$, there is a horizontal asymptote at $y = \frac{1}{1} = 1$. We have

$$\lim_{x \to -\infty} f(x) = 1, \quad \lim_{x \to \infty} f(x) = 1.$$

First, we factor, getting

$$f(x) = \frac{(x-2)(x+1)}{(x-1)(x+1)} = \frac{x-2}{x-1}.$$

Once we cancel, we see the denominator is 0 only when $x = 1$, which is where the vertical asymptote is. We have

$$\lim_{x \to 1^-} f(x) \text{ DNE } (+\infty), \quad \lim_{x \to 1^+} f(x) \text{ DNE } (-\infty).$$

2. The vertical asymptote is at $x = -1$. So we make a brief chart getting close to -1 from the left and from the right.

x	$f(x)$
-1.01	67.0044
-1.001	667.000
-0.99	-66.3378
-0.999	-666.334

Table 6: Looking at $\lim_{x \to -1^-} f(x)$ and $\lim_{x \to -1^+} f(x)$.

So from the left, we are tending toward $(+\infty)$, and from the right, we are tending toward $(-\infty)$.

3. We make a brief chart with values of x to the left and to the right.

x	$f(x)$
-100	1.0099
-1000	1.001
100	0.989899
1000	0.998999

Table 7: Looking at $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.

So as $x \to -\infty$, we see the values are a little bigger than 1, and so the graph approaches the asymptote from above. As $x \to \infty$, we see the values are a little smaller than 1, and so the graph approaches the asymptote from below.