Intermediate Value Theorem

We began our discussion of continuity by looking at the behavior of the graph of a function at certain points, such as a jump in the graph. We also needed continuity for the Extreme Value Theorem – necessary for optimization, an important application of calculus. We'll see another use for continuity – showing that equations have solutions in a given interval.

Let's start by looking at part of graph, shown in Figure 1.

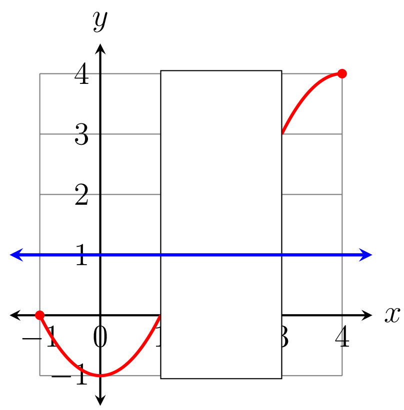

Figure 1: Illustrating the Intermediate Value Theorem.

This is the graph of some function on the closed interval $[-1, 4]$, with part of the graph obscured. Will the graph cross the blue line $y = 1$?

Not necessarily. In Figure 2, we see a simple way to complete the graph so it does *not* cross the line.

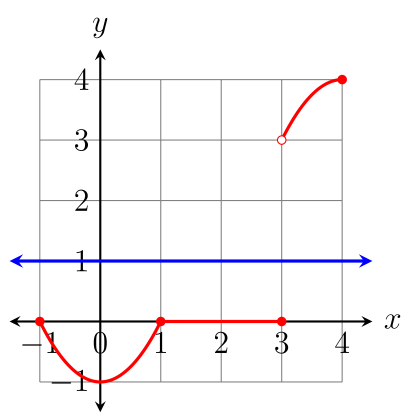

Figure 2: Illustrating the Intermediate Value Theorem.

Now let's see where continuity comes into play. Let's ask basically the same question, with a twist: if we assume that the graph in Figure 1 is a *continuous* function, does it have to cross the blue line?

The answer to this questions is "Yes." This is called the Intermediate Value Theorem in calculus. The term "intermediate" is used in the following sense: if we know two different y-values that a continuous function takes on, then it must also take on every y-value between these two values – that is, every intermediate y-value. It is usually stated as follows.

> Suppose $f(x)$ is a continuous function defined on a closed interval $[a, b]$. If $f(a) \neq f(b)$, and if c is between $f(a)$ and $f(b)$, then there is some x_0 in the open interval (a, b) such that $f(x_0) = c$.

This theorem is a way – using calculus terminology – to describe what we observed by looking at graphs. We need this terminology because there is an *infinite* number of ways of drawing continuous graphs between points – and there is no way we can draw an infinite number of graphs and look at them all. But we *can* create a proof using calculus concepts. We won't look at a proof, but it's important to know how to describe using calculus terminology what we visually observe. So let's talk through this theorem using the simplest way of creating a continuous graph – just drawing a straight line through the missing part of the graph.

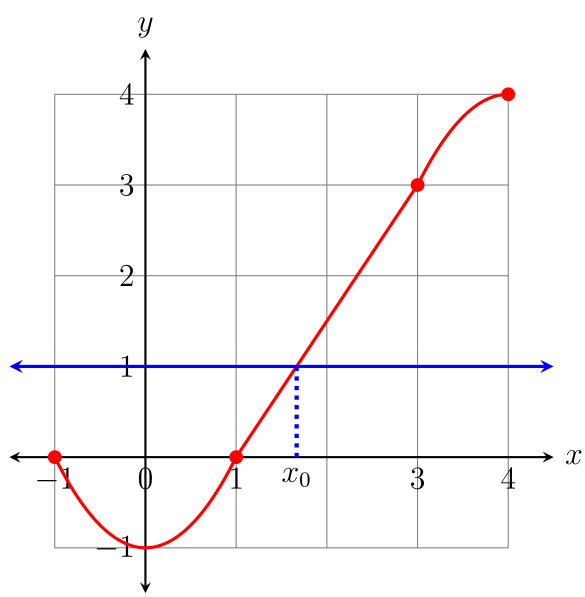

Figure 3: Illustrating the Intermediate Value Theorem.

This function is continuous – no jumps, no discontinuities. The closed interval we're looking at is $[a, b] = [-1, 4]$, as we see from the graph. We also observe that $f(-1) = 0$ and $f(4) = 4$, and so $f(-1) \neq f(4)$. Also, $c = 1$ is between 0 and 4. So $x_0 = \dfrac{5}{3}$ is that number in the open interval $(-1, 4)$ such that $f\left(\dfrac{5}{3}\right) = 1$. In Figure 4, you can see all these values annotated.

The important point is that the Intermediate Value Theorem doesn't tell you *how* to find this x_0 – it is often quite difficult, requiring a computer or calculator to find. In mathematics, we call this an **existence proof.** It tells you there *is* a thing, but not how to *find* it.

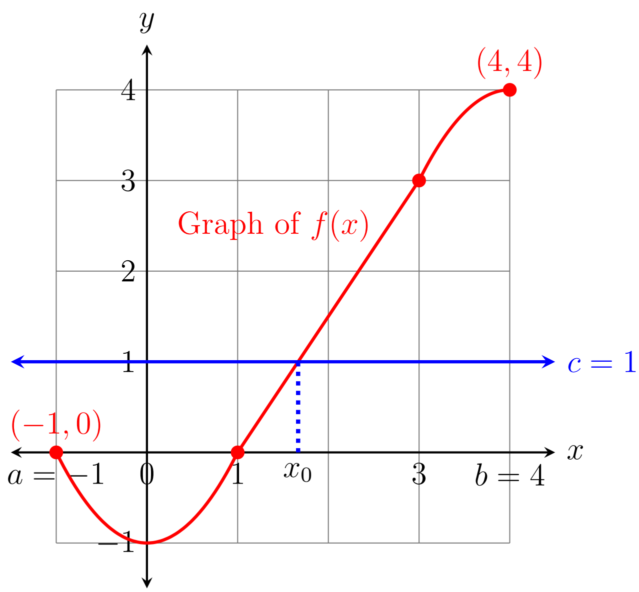

Figure 4: Illustrating the Intermediate Value Theorem.

This type of thing happens every day. Think about the stock market. Suppose you have $1000 to invest. When the stock market opens, you'd like to buy the stock which will have the highest gain at the end of the day. Which one should you buy?

Well, there is one. When the stock market closes, you'd be able to figure it out by looking at the percentage increase or decrease of all the stocks. So when the market opens, there *exists* a stock which will have the highest percentage gain, but at that time, you don't know what it is. You just know that one exists.

This is similar to the Intermediate Value Theorem. It tells you that something *exists*, but it doesn't tell you what it is. Let's look at some examples. We'll use the abbreviation IVT for the Intermediate Value Theorem.

Example 1

Show that the graphs of $y = x$ and $y = \cos(x)$ intersect somewhere in the interval $[0, \pi]$, as shown in Figure 5.

How can we use the IVT to show this? We want to show that $x = \cos(x)$ has a solution in the closed interval $[0, \pi]$. The first step is to define a function

$$f(x) = x - \cos(x).$$

Then observe that solving $x = \cos(x)$ is the same as solving $f(x) = 0$. We need a *function* to apply the IVT, so we need to create one. Let's see why this works. We must *also* state that $f(x)$ is continuous as it is a difference of two continuous functions: a basic trigonometric function and a polynomial.

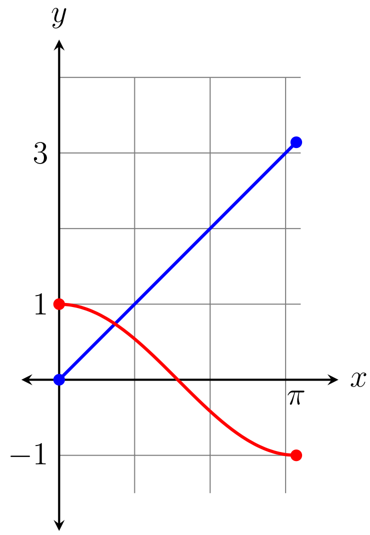

Figure 5: Graphs of $y = x$ and $y = \cos(x)$ on the interval $[0, \pi]$.

To use the IVT, we need to evaluate $f(x)$ at the endpoints. So

$$f(0) = 0 - \cos(0) = -1, \quad f(\pi) = \pi - \cos(\pi) = \pi + 1 \approx 4.14.$$

We remark that $f(0)$ is negative because at $x = 0$, the graph of $y = x$ is *below* the graph of $y = \cos(x)$, and $f(\pi)$ is positive because at $x = \pi$, the graph of $y = x$ is *above* the graph of $y = \cos(x)$.

Now the IVT states that "If $f(a) \neq f(b)$, and if c is between $f(a)$ and $f(b)$, then there is some x_0 in the open interval (a, b) such that $f(x_0) = c$." Here, $a = 0$ and $b = \pi$. Our calculations show that $f(0) \neq f(\pi)$. We'll choose $c = 0$ (since we want to solve $f(x) = 0$), and clearly 0 is between -1 and 4.14. So there must be some x_0 in $[0, \pi]$ with $f(x_0) = 0$. This means that

$$\begin{aligned} f(x_0) &= 0 \\ x_0 - \cos(x_0) &= 0 \\ x_0 &= \cos(x_0) \end{aligned}$$

Thus, the point x_0 is a point where the graphs of $y = x$ and $y = \cos(x)$ intersect. Two important points:

1. The IVT doesn't tell you *where* they intersect. But using a computer, you can approximate the solution to be $x_0 \approx 0.739$. In general, you need a computer to solve equations which combine trigonometric functions and polynomials.

2. There may be *more* than one point of intersection. The IVT tells you that a point *exists,* but doesn't tell you *how many.* To say that such a point exists means there is *at least one* point. There could be more.

It seems obvious that the graphs intersect by looking at them. Keep in mind that the IVT was proved long before the age of computers – you couldn't just type in the equations and have the graphs pop up. Typically, mathematicians use graphs to look for different features of a graph, and use results like the IVT to *prove* that these features do in fact exist.

Here's a summary of the steps to take to show that two curves intersect.

To show that two curves intersect over a given closed interval:

1. Create a function which is the difference of the equations for the curves.
2. Write a short sentence showing this function is continuous.
3. Evaluate the function at the endpoints of the closed interval.
4. State that 0 is between these values, so by the IVT, there is some x_0 where the function evaluates to 0.
5. Conclude that the curves intersect at this value x_0.

Example 2

Let's do another example where we use these steps.

Show that the graphs of $y = 4 - x$ and $y = \ln x$ intersect somewhere in the closed interval $[1, 5]$.

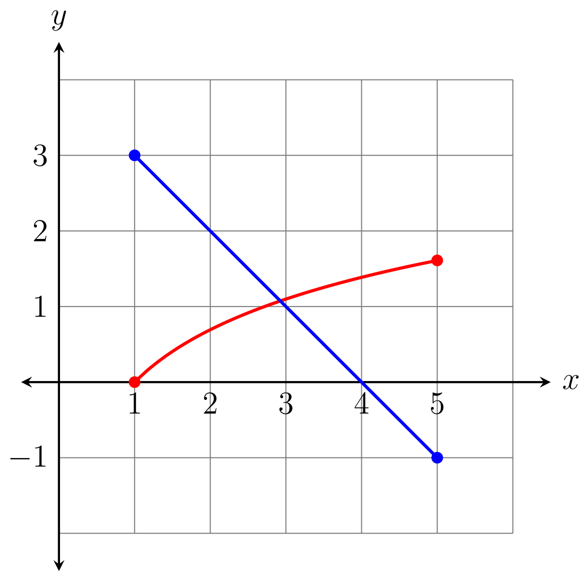

Figure 6: Graphs of $y = 4 - x$ and $y = \ln x$ on the interval $[1, 5]$.

1. Define $f(x) = 4 - x - \ln(x)$. When you subtract the functions, it doesn't make sure which order you subtract them in – the logic is the same either way.
2. Since $4 - x$ is a polynomial and $\ln x$ is continuous where it is defined, their difference is continuous.
3. We calculate:

$$f(x) = 4 - 1 - \ln 1 = 3, \quad f(5) = 4 - 5 - \ln 5 \approx -2.61.$$

4. Since 0 is between 3 and -2.61, by the IVT there is some value of x_0 between 1 and 5 such that $f(x_0) = 0$.
5. Since $f(x)$ was defined as the difference between the equations for the graphs, they intersect at this point x_0. You need a calculator or computer to find that $x_0 \approx 2.93$.

Homework

1. Show that the graphs of the curves $y = e^{2x}$ and $y = 4 - x^2$ intersect in the closed interval $[0, 2]$.
2. Show that the graphs of the curves $y = \dfrac{\ln x}{x}$ and $y = e^x - 5$ intersect in the closed interval $[1, 2]$.
3. Show that the graphs of the curves $y = \sin(2x)$ and $y = 1 - \cos(x)$ intersect three times in the closed interval $[0, 2\pi]$. Hint: Graph these functions on desmos. You should only have to use the IVT once, but you'll see that you can't use the closed interval $[0, 2\pi]$ when you apply the IVT.

Solutions

Problem 1

1. Let $f(x) = e^{2x} - (4 - x^2) = e^{2x} - 4 + x^2$.
2. $f(x)$ is a difference of an exponential function and a polynomial, both of which are continuous.
3. Evaluate:
$$f(0) = -3, \quad f(2) \approx 54.6.$$
4. Note that $-3 < 0 < 54.6$, so by the IVT, there is some x_0 where $f(x_0) = 0$.
5. This means that the curves intersect at $(x_0, f(x_0))$ (and possibly other points).

Problem 2

1. Let $f(x) = \dfrac{\ln x}{x} - (e^x - 5) = \dfrac{\ln x}{x} - e^x + 5$.
2. Since $y = \ln x$ and $y = x$ are both continuous, so is their quotient. Subtracting an exponential function and a polynomial function gives a continuous function. (Not necessary to include, but note we cannot ever divide by 0 since 0 is not in the closed interval $[1, 2]$).
3. Evaluate at the endpoints.
$$f(1) \approx 2.28, \quad f(2) \approx -2.02.$$
4. Note that $-2.02 < 0 < 2.28$, so by the IVT, there is some x_0 where $f(x_0) = 0$.
5. This means that the curves intersect at $(x_0, f(x_0))$ (and possibly other points).

Problem 3

1. Let $f(x) = \sin(2x) - (1 - \cos(x)) = \sin(2x) - 1 + \cos(x)$.
2. Since basic trigonometric functions and $y = 2x$ and $y = -1$ (polynomials) are continuous functions, so are compositions, sums, and differences of these functions.
3. By looking at the graphs on desmos, it appears that the curves intersect at $x = 0$ and $x = 2\pi$. It is not difficult to evaluate $f(0) = 0$ and $f(2\pi) = 0$, meaning that the curves intersect at $x = 0$ and $x = 2\pi$. It looks like there is a third intersection somewhere between $\dfrac{\pi}{4}$ and $\dfrac{\pi}{2}$, so we use the closed interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$. We now evaluate at the endpoints of this interval.
$$f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) - 1 + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad f\left(\frac{\pi}{2}\right) = \sin(\pi) - 1 + \cos\left(\frac{\pi}{2}\right) = -1.$$

4. Note that $-1 < 0 < \frac{1}{\sqrt{2}}$, so by the IVT, there is some x_0 where $f(x_0) = 0$.
5. This means that the curves intersect at $(x_0, f(x_0))$ (and possibly other points).