Applications of the Extreme Value Theorem

Example 1

Suppose two positive numbers sum to 10. What is the largest their product can be?

We can make a chart to try and guess the answer.

First #	Second #	Product
1	9	9
2	8	16
3	7	21
4	6	24
4.5	5.5	24.75
5	5	25

Of course there's no need to continue the chart since we'll just get the same numbers again, though in the opposite order. Also, the numbers don't always have to be integers.

Making a chart is not a rigorous mathematical justification. We'll use what we learned about optimization to show that the largest the product can be is in fact, 25.

Notice that we are not given a function in this problem. This is what makes optimization problems tricky. You have to work out what the function is *before* you start using calculus.

We'll use the notation $f(x)$, so we need to decide just what "x" represents – just like t represented time when we looked at displacement and velocity graphs. Since we're looking for the product of two numbers, we can represent the first number by x.

We might be tempted to say, "Well, let y represent the second number. That way, we can represent the product by $f(x) = x \cdot y$." The only problem with this is that now we have *two* variables – but optimizing with two variables is much more difficult, and doesn't come until Calculus III.

But we're given a bit more information. We know that the sum of the two numbers has to be 10, which we can write as

$$x + y = 10.$$

Now we can solve this for y, which gives up $y = 10 - x$. Plugging back in, we can rewrite our function as

$$f(x) = x(10 - x).$$

Now we have our function. The important observation is that we are asked for the largest product, which means we need a *global maximum*. We just learned a way to find global extrema for continuous functions – and $f(x)$ is a polynomial, so it is continuous.

Another tricky part, though, is the Extreme Value Theorem can be used *only* when the function is defined on a closed interval. So not only do we need to determine a function, we need a reasonable interval to define the function on.

Since the numbers have to be positive, it makes sense to start at 0. What is the upper limit? Well, since the positive numbers have to sum to 10, there is no way one of the numbers can be larger than 10. So a closed interval which makes sense is $[0, 10]$.

What we've done is "translated" the original word problem into an optimization problem: Find the global extrema of the function $f(x) = x(10 - x)$ on the closed interval $[0, 10]$. Let's proceed to apply the three steps to finding these extrema. Keep in mind that we are not asked for a *minimum* value, so we really only have to look for a global maximum in this example.

1. First, find out where $f'(x) = 0$. Note that it is simpler to multiply out $f(x)$ instead of trying to use the Product Rule right away.

$$\begin{aligned} f(x) &= x(10 - x) \\ &= 10x - x^2 \\ f'(x) &= 10 - 2x = 0 \\ x &= 5 \end{aligned}$$

2. Next, evaluate $f(x)$ at these points as well as the endpoints.

$$f(5) = 25, \quad f(0) = 0, \quad f(10) = 0.$$

3. By looking at the values just obtained, we see that there is a global maximum at the point $(5, 25)$, and so 25 is largest possible product.

This seems like a lot of work for just one problem, but it is important to understand *why* we need to take each step. Once we've done it once, we can summarize the process and use it to investigate more examples.

Optimization Strategy

> To solve an optimization word problem:
>
> 1. Determine what the variable x represents, and *write it down.*
> 2. Use this to find a function $f(x)$ to optimize. Sometimes it will look like you need two variables (like in the previous example), but you will *always* be able to break it down to just one variable.
> 3. Find a closed interval which makes sense for the problem.
> 4. Find the global extrema (whichever you are asked for) using the Extreme Value Theorem.

The main challenge here is that there is no one-size-fits-all method to complete steps (1), (2), and (3). It will be different for each problem. Once we're at step (4), we can use what we learned about the Extreme Value Theorem. This is often the easiest part. So we'll look at some more examples to see how to set up various types of word problems.

Example 2

Suppose you are given a positive number. First, take the square root. Then add 3. Finally, subtract the given number. What is the largest number you can obtain as a result?

Let's look at an example to see what is being asked. If we start with 4, we get $\sqrt{4} = 2$. Then we add 3, giving $2 + 3 = 5$. Finally, subtract 4 (the original number) to get $5 - 4 = 1$.

It turns out we can do a bit better than that. We'll find out how much better by using optimization.

So let's go through the steps one by one.

1. It makes sense to let x represent the positive number you are given.
2. What are we asked to optimize? First we take a number and take the square root, $\sqrt{x}$. Then add $3 : \sqrt{x} + 3$. Finally subtract the *given number,* x, giving $\sqrt{x} + 3 - x$. So

$$f(x) = \sqrt{x} + 3 - x.$$

   Note that in this case, there is no need to introduce a second variable.

3. Since x represents a positive number, it makes sense to start at 0. Note that as x gets larger, it becomes greater than $\sqrt{x}$, and is subtracted from $\sqrt{x}$. So $f(x)$ will eventually become negative. When? We know that $f(4) = 1$, which is still positive. But

$$f(9) = \sqrt{9} + 3 - 9 = -3,$$

so it looks like we can stop at $x = 9$, giving the closed interval $[0, 9]$.

4. So now we have the optimization problem of finding the global maximum of $f(x)$ on the closed interval $[0, 9]$. We have a three-step process to do this.

   (a) When is $f'(x) = 0$?

$$\begin{aligned} f(x) &= \sqrt{x} + 3 - x \\ f'(x) &= \frac{1}{2\sqrt{x}} - 1 = 0 \\ \frac{1}{2\sqrt{x}} &= 1 \\ 2\sqrt{x} &= 1 \\ \sqrt{x} &= \frac{1}{2} \\ x &= \frac{1}{4} \end{aligned}$$

   (b) Now evaluate $f(x)$ at this value and the endpoints.

$$f\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} + 3 - \frac{1}{4} = \frac{1}{2} + 3 - \frac{1}{4} = \frac{13}{4}, \quad f(0) = 3, \quad f(9) = -3.$$

   (c) The largest of these values is $\frac{13}{4}$, so there is a global maximum at $\left(\frac{1}{4}, \frac{13}{4}\right)$. Note that it would be very difficult to guess this value just by making a chart, so we really do need to use calculus here.

Example 3

Now we'll move on to some examples from geometry. Suppose you want to fence in a rectangular area next to a wall, as shown in Figure 1. If you have 60 m of fencing, what is the largest area you can enclose?

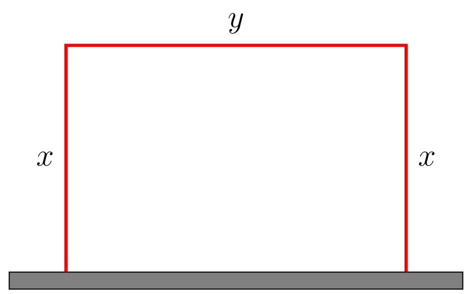

Figure 1: A fenced-in area next to a wall.

1. Let x represent the length of one of the sides of the rectangle, and y the length of the other side, as labelled in Figure 1. It is important to note that you could have labeled the horizontal side of the rectangle x and the vertical sides y, and you would still get the same answer. The algebra would be different – but you'd still get the same answer. It is often possible to make more than one model for the same word problem.

2. Since we are asked to maximize the area, it makes sense to let $f(x) = x \cdot y$, which is the formula for the area of a rectangle. This gives us two variables again – but since we know we have 60 m of fencing, we know that

   $$x + y + x = 60.$$

   From this, we get $y = 60 - 2x$, so

   $$f(x) = x(60 - 2x) = 60x - 2x^2.$$

   Again, it's easier to multiply out so we can use the Power Rule instead of the Product Rule.

3. What closed interval should we choose? Since we have 60 m of fence total, no side can be less that 0 m or greater than 60 m, so $[0, 60]$ would be one choice. But if you see that *two* of the sides of the rectangle have length x, then x cannot be greater than 30 m. So you can also use $[0, 30]$. It's usually easier to work with smaller numbers, so let's use $[0, 30]$.

4. Now we've translated the word problem into the following optimization problem: Find the global maximum of the function $f(x) = 60x - 2x^2$ on the closed interval $[0, 30]$. So we now follow the three steps for solving an optimization problem.

(a) Observe that $f'(x)$ always exists, since $f(x)$ is a polynomial. We need to find where $f'(x) = 0$.

$$\begin{aligned} f(x) &= 60x - 2x^2 \\ f'(x) &= 60 - 4x = 0 \\ 4x &= 60 \\ x &= 15 \end{aligned}$$

(b) Now evaluate at this point and the endpoints.

$$f(0) = 0, \quad f(15) = 450, \quad f(30) = 0.$$

(c) The largest value is 450, so there is a global maximum at $(15, 450)$, and thus the largest possible area is 450 m^2.

Example 4

Suppose you are given a right isosceles triangle whose legs are 2 units long. Inscribe a rectangle in the triangle as shown in Figure 2. What is the largest the area of such a rectangle can be?

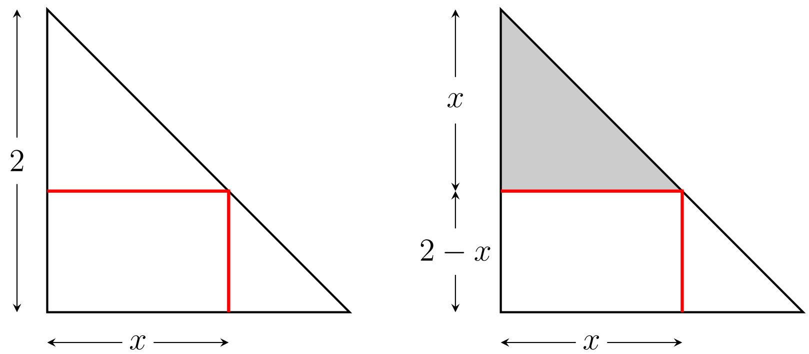

Figure 2: Optimizing the area of a rectangle inscribed in a triangle.

1. Let x represent the width of the rectangle, as shown in Figure 2.

2. Since we are looking to maximize the area of the rectangle, we need a function to represent length $\times$ width. We called x the width – so what is the height? Look at the shaded right isosceles triangle on the right of Figure 2. Since the horizontal leg has length x, then the vertical leg has length x as well. So to get the height of the rectangle, we just substract x from 2 to get $2 - x$. Thus, the area of the rectangle is $f(x) = x(2 - x) = 2x - x^2$.

3. Looking at how the rectangle is inscribed in the triangle, we see that an appropriate closed interval for x is $[0, 2]$.

4. Now we have the following optimization problem: find the global maximum of $f(x) = 2x - x^2$ on the closed interval $[0, 2]$.

   (a) First, note that $f'(x)$ exists everywhere since $f(x)$ is a polynomial. Now we need to see where $f'(x) = 0$.

   $$\begin{aligned} f(x) &= 2x - x^2 \\ f'(x) &= 2 - 2x = 0 \\ 2x &= 2 \\ x &= 1 \end{aligned}$$

   (b) Now evaluate $f(x)$ at this point and the endpoints.

   $$f(0) = 0, \quad f(1) = 1, \quad f(2) = 0.$$

   (c) Note that 1 is the largest among these values, so there is a global maximum at $(1, 1)$. Thus, the largest possible area is 1 unit^2, in which case the rectangle is actually a square.

Example 5

Now let's look at an example in three dimensions. Suppose you want to make a cylindrical steel can which will hold 250π cm^3 (which is a little more than 26 ounces). How can you do this using the least amount of steel?

We'll need a few formulas from geometry. If a cylinder has a radius of r and a height h, then its volume is $V = \pi r^2 h$ and its surface area is $S = 2\pi r^2 + 2\pi r h$. Remember, the units of V are cm^3 and the units of S are cm^2.

Because we are minimizing how much steel we are using, we want to minimize the surface area formula, $S = 2\pi r^2 + 2\pi r h$. This equation has two variables, so we need some way to eliminate one of them. But we are also given that the volume is 250π cm^3, which means

$$\begin{aligned} \pi r^2 h &= 250\pi \\ r^2 h &= 250 \end{aligned}$$

So have a choice here: solve for r, or solve for h. Which one should we pick? You can see that if we try to solve for r, we'll need to take a square root – and equations involving square roots are usually more difficult to work with. So let's solve for h. It is worth pointing out that if you solve for r, you will still be able to get the correct answer, but it will take a little more work.

$$\begin{aligned} r^2 h &= 250 \\ h &= \frac{250}{r^2} \end{aligned}$$

Now we substitute this back into the surface area formula. This will give us a formula where r is the only variable.

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r h \\ S(r) &= 2\pi r^2 + 2\pi r \cdot \frac{250}{r^2} \\ S(r) &= 2\pi r^2 + \frac{500\pi}{r} \\ &= 2\pi r^2 + 500\pi r^{-1} \end{aligned}$$

Now that we have an equation to optimize, we need to find a suitable closed interval. Of course the radius has to be positive, so we might be tempted to use 0 as the left endpoint. But $S(0)$ is undefined, since there would be a 0 in the denominator. So we'll choose a small value of r as the left endpoint, say 0.1.

We can "guesstimate" by observing that if r was pretty big – say 20 cm, then h would be less than 1 cm, and we'd have a very short can with a very large top and bottom ($r = 20$

cm would be a can with top and bottom over a foot in diameter). So 20 is a safe guess for the right endpoint. Therefore, we've translated our word problem into the following optimization problem: find the global minimum of $S(r) = 2\pi r^2 + 500\pi r^{-1}$ over the closed interval $[0.1, 20]$.

Now all that's left is to apply the Extreme Value Theorem.

1. First, determine where $S'(r) = 0$.

$$\begin{aligned} S(r) &= 2\pi r^2 + 500\pi r^{-1} \\ S'(r) &= 2\pi \cdot 2r + 500\pi(-1)r^{-2} \\ &= 4\pi r - \frac{500\pi}{r^2} = 0 \\ 4\pi r &= \frac{500\pi}{r^2} \\ 4\pi r^3 &= 500\pi \\ r^3 &= 125 \\ r &= 5 \end{aligned}$$

2. Next, evaluate $S(r)$ at this point and the endpoints.

$$S(0.1) \approx 15708, \quad S(5) \approx 471.24, \quad S(20) \approx 2591.8.$$

3. Looking at the smallest value, we see that we use the least amount of steel when the radius is 5 cm, and that we use about 471.24 cm^2 of steel.

Homework:

1. The product of two positive numbers is 16. What is the smallest possible value for their sum?

2. Suppose you start with a positive number. Square it, and then multiply by 2. Then subtract the square root of the given number. What is the smallest result you can obtain by doing this? Note: the answer is a simple fraction.

3. Suppose you have 40 m of fencing and you want to enclose a rectangular region in a corner, as shown in Figure 3. What is the largest area you can enclose?

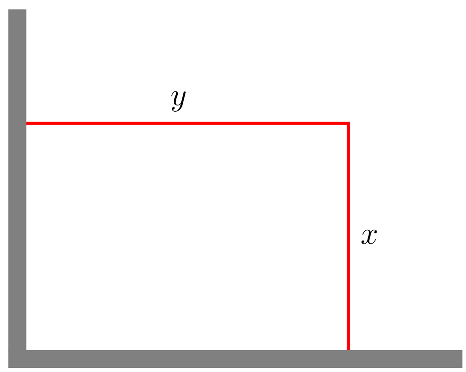

Figure 3: A fenced-in area in a corner.

4. Suppose you want to inscribe a rectangle in a right triangle, as shown in Figure 4. What is the largest area of such a rectangle? Hint: you will need to look at ratios of corresponding sides of similar triangles for this problem.

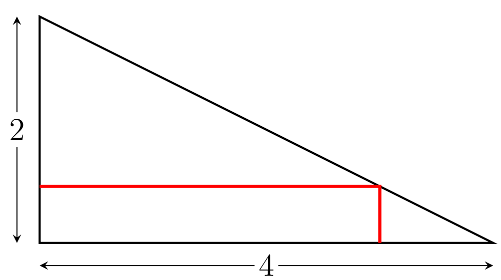

Figure 4: Optimizing the area of a rectangle inscribed in a triangle.

5. Suppose you want to make a cylindrical steel can with an open top which will hold 1000π cm^3. How can you do this using the least amount of steel? Note: you will have to slightly modify the surface area formula by subtracting the area of the top of the can.

Solutions

Problem 1

1. Let x be one of the positive numbers, and let y be the other one.

2. Since we are minimizing the sum, let $f(x) = x + y$. We are given that $xy = 16$, and so $y = \dfrac{16}{x}$, so that
$$f(x) = x + \frac{16}{x}.$$

3. Since x must be positive but cannot be 0, we choose a small value like 0.1 for the left endpoint. For the right endpoint, we "guesstimate." Now $16 \cdot 1 = 16$ and $16 + 1 = 17$, while $8 \cdot 2 = 16$ and $8 + 2 = 10$. So it looks like the sums getting larger as we from $x = 8$ to 16 and beyond. So we'll choose the interval $[0.1, 16]$.

4. Now optimize $f(x) = x + \dfrac{16}{x}$ on the closed interval $[0.1, 16]$.

   (a) Then $f(x) = x + 16x^{-1}$, so that $f'(x) = 1 - 16x^{-2}$. Note that $f'(x)$ is always defined (remember, x cannot be 0 since it is not in the interval $[0.1, 16]$).
$$\begin{aligned} f'(x) &= 0 \\ 1 - \frac{16}{x^2} &= 0 \\ 1 &= \frac{16}{x^2} \\ x^2 &= 16 \\ x &= -4, +4 \\ x &= 4 && -4 \text{ is not in the domain} \end{aligned}$$

   (b) Now evaluate at this point and the endpoints.
$$f(0.1) = 160.1, \quad f(4) = 8, \quad f(16) = 17.$$

   (c) The smallest of these values is 8, so 8 is the minimum possible sum.

Problem 2

1. Let x represent the positive number.
2. $f(x) = x^2 \cdot 2 - \sqrt{x} = 2x^2 - x^{1/2}$.
3. $f(0)$ is defined, so we take 0 to be the left endpoint. Let's try a few more values that are easy to calculate:

$$f(4) = 2 \cdot 16 - 2 = 30, \quad f(9) = 2 \cdot 81 - 3 = 159.$$

   We'll only get bigger after $x = 9$, so we'll choose our closed interval to be $[0, 9]$.

4. Now optimize $f(x) = 2x^2 - x^{1/2}$ on the closed interval $[0, 9]$. Note that

$$f'(x) = 4x - \frac{1}{2}x^{-1/2} = 4x - \frac{1}{2\sqrt{x}}.$$

   (a) $f'(x)$ is undefined at $x = 0$, but this is an endpoint, so it's already taken care of. Now solve $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ 4x - \frac{1}{2\sqrt{x}} &= 0 \\ 4x &= \frac{1}{2\sqrt{x}} \\ 4x \cdot 2\sqrt{x} &= 1 \\ x^{3/2} &= \frac{1}{8} \\ \left(x^{3/2}\right)^{2/3} &= \left(\frac{1}{8}\right)^{2/3} \\ x &= \frac{1}{4} \end{aligned}$$

   (b) Evaluate here and at the endpoints.

$$f(0) = 0, \quad f\left(\frac{1}{4}\right) = -\frac{3}{8}, \quad f(9) = 159.$$

   (c) We conclude that $-\frac{3}{8}$ is the smallest value for $f(x)$, obtained at $x = \frac{1}{4}$.

Problem 3

1. Let x represent the length of one of the sides of the rectangle, and y the length of the other side.

2. Since we are asked to maximize the area, it makes sense to let $f(x) = x \cdot y$. This gives us two variables again – but since we know we have 40 m of fencing, we know that

$$x + y = 40.$$

From this, we get $y = 40 - x$, so

$$f(x) = x(40 - x) = 40x - x^2.$$

3. Since we have 40 m of fence total, no side can be less that 0 m or greater than 40 m, so $[0, 40]$ would be a good choice.

4. Now we've translated the word problem into the following optimization problem: Find the global maximum of the function $f(x) = 40x - x^2$ on the closed interval $[0, 40]$.

   (a) Observe that $f'(x)$ always exists, since $f(x)$ is a polynomial. We need to find where $f'(x) = 0$.

$$\begin{aligned} f(x) &= 40x - x^2 \\ f'(x) &= 40 - 2x = 0 \\ 2x &= 40 \\ x &= 20 \end{aligned}$$

   (b) Now evaluate at this point and the endpoints.

$$f(0) = 0, \quad f(20) = 400, \quad f(40) = 0.$$

   (c) The largest value is 400, so there is a global maximum at $(20, 400)$, and thus the largest possible area is 400 m^2.

Problem 4

1. Let x be the width of the rectangle, as shown in Figure 5.

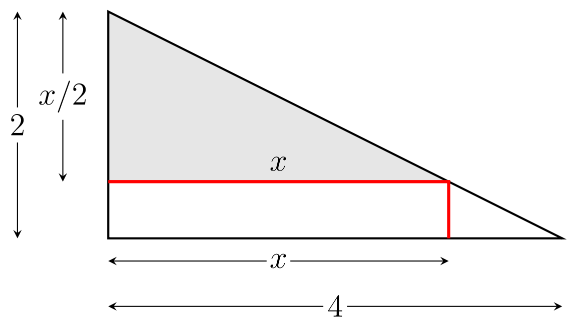

Figure 5: Optimizing the area of a rectangle inscribed in a triangle.

2. In the original triangle, the longer leg is twice the shorter leg, and so in the similar gray shaded triangle, the shorter leg must have length $x/2$. This means the height of the rectangle is $2 - \dfrac{x}{2}$. We are maximizing the area, so we let $f(x)$ be the width times the height, or
$$f(x) = x\left(2 - \frac{x}{2}\right) = 2x - \frac{x^2}{2}.$$

3. Looking at how the rectangle is inscribed in the triangle, we see that an appropriate closed interval for x is $[0, 4]$.

4. Now we have the following optimization problem: find the global maximum of $f(x) = 2x - \dfrac{x^2}{2}$ on the closed interval $[0, 4]$.

   (a) First, note that $f'(x)$ exists everywhere since $f(x)$ is a polynomial. Now we need to see where $f'(x) = 0$.
   $$\begin{aligned} f(x) &= 2x - \frac{x^2}{2} \\ f'(x) &= 2 - x = 0 \\ x &= 2 \end{aligned}$$

   (b) Now evaluate $f(x)$ at this point and the endpoints.
   $$f(0) = 0, \quad f(2) = 2, \quad f(4) = 0.$$

   (c) Note that 2 is the largest among these values, so there is a global maximum at $(2, 2)$. Thus, the largest possible area is 2 unit^2.

Problem 5

Example 5

1. Let r be the radius of the cylinder and h the height.
2. We are optimizing the surface area of a cylinder without a top. Since the top is a circle of area πr^2, our surface area this time is
$$S = 2\pi r^2 + 2\pi rh - \pi r^2 = \pi r^2 + 2\pi rh.$$
We are given a volume of 1000π cm^3, and so
$$\begin{aligned} \pi r^2 h &= 1000\pi \\ r^2 h &= 1000 \\ h &= \frac{1000}{r^2} \end{aligned}$$
Now we substitute this back into the surface area formula. This will give us a formula where r is the only variable.
$$\begin{aligned} S &= \pi r^2 + 2\pi rh \\ S(r) &= \pi r^2 + 2\pi r \cdot \frac{1000}{r^2} \\ S(r) &= \pi r^2 + \frac{2000\pi}{r} \\ &= \pi r^2 + 2000\pi r^{-1} \end{aligned}$$
3. Since the function is not defined at $r = 0$, we choose a left endpoint to be 0.1, as in Example 5. Since the problem is so similar to Example 5, we choose $r = 20$ as the right endpoint.
4. We've translated our word problem into the following optimization problem: find the global minimum of $S(r) = \pi r^2 + 2000\pi r^{-1}$ over the closed interval $[0.1, 20]$.

   Now all that's left is to apply the Extreme Value Theorem.

   (a) First, determine where $S'(r) = 0$.
$$\begin{aligned} S(r) &= \pi r^2 + 2000\pi r^{-1} \\ S'(r) &= \pi \cdot 2r + 2000\pi(-1)r^{-2} \\ &= 2\pi r - \frac{2000\pi}{r^2} = 0 \\ 2\pi r &= \frac{2000\pi}{r^2} \\ 2\pi r^3 &= 2000\pi \\ r^3 &= 1000 \\ r &= 10 \end{aligned}$$

(b) Next, evaluate $S(r)$ at this point and the endpoints.

$$S(0.1) \approx 62832, \quad S(10) \approx 942.48, \quad S(20) \approx 1570.8.$$

(c) Looking at the smallest value, we see that we use the least amount of steel when the radius is 10 cm, and that we use about 942.48 cm^2 of steel.