# Optimization

In calculus, the term **optimization** involves finding mimima or maxima of a function. For example, when is the tide the highest? What price will maximize profit? How can you build a box using the least amount of wood? These are questions of optimization.

First, we need a little terminology. We've used the terms "minimum" and "maximum" informally, but now we need to be a little more precise. Let's look at the graph in Figure 1.

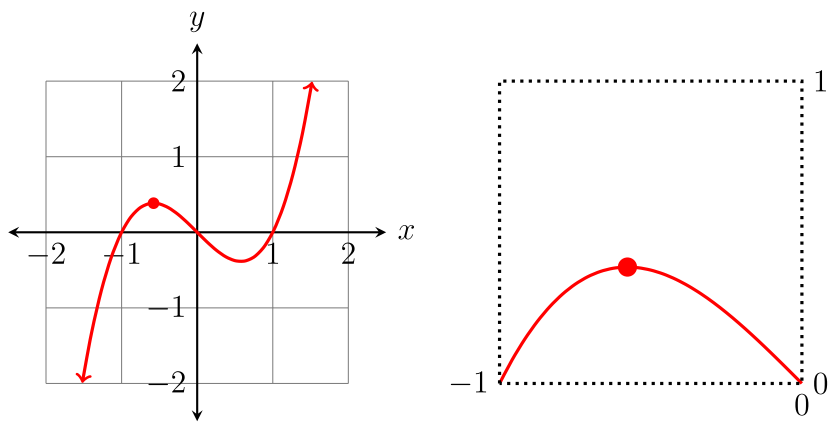



Figure 1: Local extrema: zooming in.

Near $-0.6$, the graph has a **local maximum.** In other words, if we zoom in closer, as on the right of Figure 1, it looks like the highest point on the graph is near $x = -0.6$. If we zoom back out (the left graph of Figure 1), we see that this point is not the highest point on *the entire graph.* So it is not a **global maximum** – the highest point on a graph.

Near $x = 0.6$, we see that these is a **local minimum** – if we zoom in, it will look like the graph has a lowest point at $x \approx 0.6$. But it is not a **global minimum,** since it is not the lowest point on *the entire graph.*

We use the term **local extremum** to mean either a local minimum or maximum, and the term **global extremum** to mean either a global minimum or global maximum. Another common term for global extremum is **absolute extremum.** You will likely see both.

| Looking for... | think... |
|---|---|
| Local extrema | Zooming in |
| Global extrema | Zooming out |

## Example 1

Consider the graph of $f(x)$ shown below. The arrows means that the graph keeps going up (it's actually a fourth-degree polynomial).

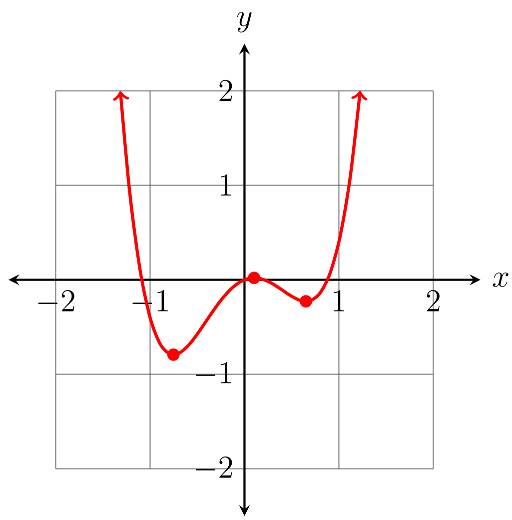



Figure 2: Local and global extrema.

Let's look at some features of this graph using our new terminology. At $x \approx -0.8$, we have a local minimum *and* a global minimum, as this point is the lowest on the *entire* graph. At $x \approx 0.7$, we have a local minimum – but it's not a global minimum because there are lower points on the graph. At $x \approx 0.1$, we have a local maximum – but it's not a global maximum since the graphs extends upward toward infinity. There is *no* global maximum on this graph.

It is worth pointing out that some graphs have *no* local or global extrema. Take the exponential function $f(x) = e^x$, for example. It is *always* increasing, so there can be no local or global maxima. There is no local or global minimum, either. You might be tempted to think that 0 is a global minimum. But it is not possible to solve $e^x = 0$, so there is *no* $x$-value that has a $y$-value of 0. So there are no minima, either.

**Example 2**

We now look at how to find extrema of a function. We consider the function $f(x) = \sin(x)$ on the interval $[0, 4\pi]$. Note that when we restrict the domain, we are looking for global extrema over the interval $[0, 4\pi]$ *only*. This is very common in mathematics and science. Often, the horizontal axis represents time, and you only ever consider some finite period of time, *not* an infinite period.

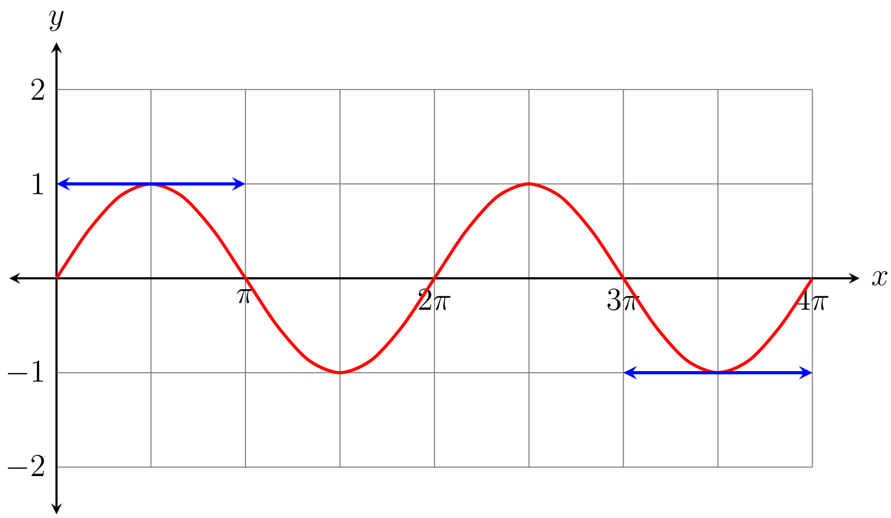



Figure 3: Graph of $f(x) = \sin(x)$ on the interval $[0, 4\pi]$.

We observed earlier that when we have a local extremum, we have a horizontal tangent, as seen in Figure 3. But a horizontal line has a slope of 0, and so 0 is the slope of the tangent line – which is given by the derivative. So our strategy should be to find out where $f'(x) = 0$.

But $f'(x) = \cos(x)$. From the unit circle, we know that solving $\cos(x) = 0$ on the interval $[0, 4\pi]$ gives four solutions:

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}.$$

We know what type of extrema these are by looking at the graph. Can we do this *without* a graph? The key observation is that at a local maximum, the graph is concave down, and at a local minimum, the graph is concave up. We determine whether a graph is concave down or up by looking at $f''(x)$. So

$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \end{aligned}$$

So

$$f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 < 0,$$

so the graph must be concave down at $x = \frac{\pi}{2}$ since the second derivative is negative there, giving a local maximum. Similarly,

$$f''\left(\frac{3\pi}{2}\right) = -\sin\left(\frac{3\pi}{2}\right) = 1 > 0,$$

so the graph must be concave up at $x = \frac{3\pi}{2}$ since the second derivative is positive there, giving a local minimum. There is also a local maximum at $x = \frac{5\pi}{2}$ and a local minimum at $x = \frac{7\pi}{2}$.

Are there any global extrema? It turns out there are *two* global extrema, $\left(\frac{\pi}{2}, 1\right)$ and $\left(\frac{5\pi}{2}, 1\right)$, since there are two values of $x$ where $f(x) = 1$ (again, restricting attention to the given domain). Likewise, there are two global minima at $x = \frac{3\pi}{2}$ and $x = \frac{7\pi}{2}$.

Two important points to take away: there may be multiple global extrema, and we can use the second derivative to help us determine if local extrema are minima or maxima.

### Example 3

In this example, we consider the function $f(x) = 1 - x^4$, shown in Figure 4.

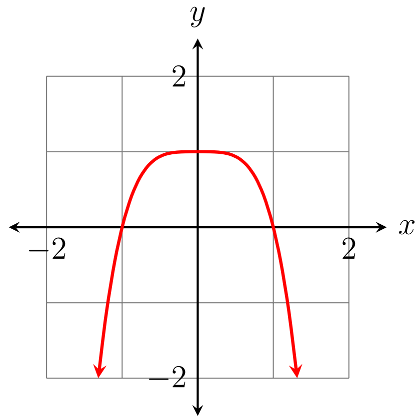



Figure 4: Graph of $f(x) = 1 - x^4$.

We can observe a local and global maximum at the point $(0, 1)$. Let's use calculus to verify this. Remember, there is a horizontal tangent there, we need to find out where $f'(x) = 0$.

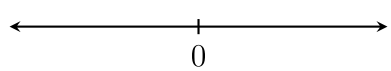
$$\begin{aligned} f(x) &= 1 - x^4 \\ f'(x) &= -4x^3 \\ -4x^3 &= 0 \\ x &= 0 \end{aligned}$$

So, as expected, we have $f'(0) = 0$. Now let's try the second derivative:

$$\begin{aligned} f'(x) &= -4x^3 \\ f''(x) &= -12x^2 \\ f''(0) &= 0 \end{aligned}$$

So since $f''(0) = 0$, we can't tell whether the graph is concave up or concave down there – there might even be an inflection point. So we need to use a sign chart (just like we did in the notes for Day 10). We'll use the three steps illustrated on p. 6 of that handout.

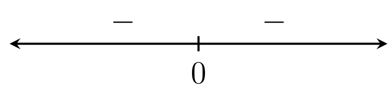
1. We already know that solving $f''(x) = 0$ gives $x = 0$.
2. This gives the following number line:



3. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$
\begin{aligned}
f''(-1) &= -12(-1)^2 \\
&= -12 \\
&< 0 \\
f''(1) &= -12(1^2) \\
&- 12 \\
&< 0.
\end{aligned}
$$

This yields the following number line:




There is no inflection point since the concavity does not change – it's concave down on both sides of $x = 0$. This means that $x = 0$ is a local maximum, and in this case, also a global maximum.

This and the last example show you that to find local extrema, we set $f'(x) = 0$. To see if the function is concave up or down, use $f''(x)$. This works *except* when $f''(x) = 0$, in which case you need to make a sign chart. Here's a summary.

> To find local extrema of $f(x)$ :
>
> 1. Determine where $f'(x) = 0$.
> 2. Find $f''(x)$ at these points.
>    (a) If $f''(x) > 0$, there is a local minimum.
>    (b) If $f''(x) < 0$, there is a local maximum.
>    (c) If $f''(x) = 0$, use a sign chart for $f''(x)$.

## Global Minima and Maxima

Not every function has local or global extrema. But in certain circumstances, we can know that *global* extrema do in fact exist.

| |
|---|
| If a function is defined on a closed interval and is continuous, both a global minimum and a global maximum exist. |

What is so important about a closed interval? Let's look at $f(x) = \dfrac{1}{x}$, shown in Figure 5.

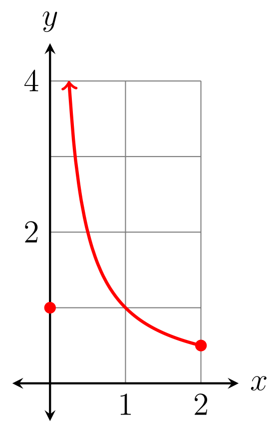



Figure 5: The importance of a closed interval.

We know what the graph looks like on $(0,2]$; there is a vertical asymptote. Now suppose we wanted to create a continuous function on $[0,2]$ by defining the function to be some value at $x = 0$. Can you see why this is impossible? No matter how we defined $f(0)$ – for example, $f(0) = 1$ – in order to be continuous at 0, the function would somehow have to turn around and come back down to the point $(0,1)$. This cannot be done if there is a vertical asymptote at $x = 0$.

Essentially, by making the assumption that the function is defined on a closed interval, it is not possible for there to be any vertical asymptotes. So there must be a lowest and highest point somewhere on the graph. A formal proof is a bit more complicated, but the graph in Figure 5 is meant to give you an idea of why this must be true.

So if we know that global extrema exist, how do we find them? There is a straightforward way using calculus. First, we'll give the method and then do some examples.

> Suppose a function $f(x)$ is defined on a closed interval $[a, b]$ and is continuous. Then both a global minimum and a global maximum exist. To find them:
>
> 1. Determine where $f'(x) = 0$ or $f'(x)$ does not exist,
> 2. Evaluate $f(x)$ at these points and the endpoints $a$ and $b$,
> 3. Select the lowest and highest values among these function values.

**Example 4**

Let the function $f(x) = 3x - x^3$ be defined on the closed interval $[-2, \sqrt{3}]$. Find the global extrema. The graph is shown in Figure 6.

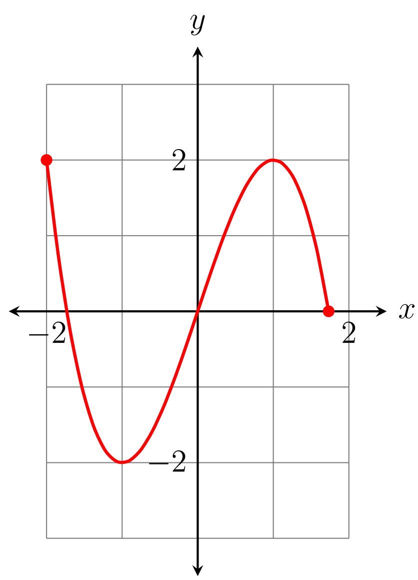



Figure 6: The graph of $f(x) = 3x - x^3$ on the closed interval $[-2, \sqrt{3}]$.

Let's proceed with the steps one by one.

1. Using the Power Rule, we get $f'(x) = 3 - 3x^2$. Since the derivative is a polynomial, it exists everywhere. To see where it's 0, we solve.

$$\begin{aligned} f'(x) &= 0 \\ 3 - 3x^2 &= 0 \\ 3 &= 3x^2 \\ x^2 &= 1 \\ x &= -1, +1 \end{aligned}$$

2. Now evaluate at these points and the endpoints. Note that we want function values here, so we plug into $f(x)$.

$$f(-1) = -2, \quad f(1) = 2, \quad f(-2) = 2, \quad f(\sqrt{3}) = 0.$$

3. Looking at these function values, $-2$ is the lowest and 2 is the highest. Thus, there is a global minimum at $(-1, -2)$, and global maxima at $(-2, 2)$ and $(1, 2)$. Of course, these results make perfect sense by looking at the graph.

**Example 5**

It's important to note that you do *not* have to assume that the derivative exists everywhere in order to find global extrema. Let's take the example of $f(x) = |x|$ on the closed interval $[-2, 1]$, shown in Figure 7.

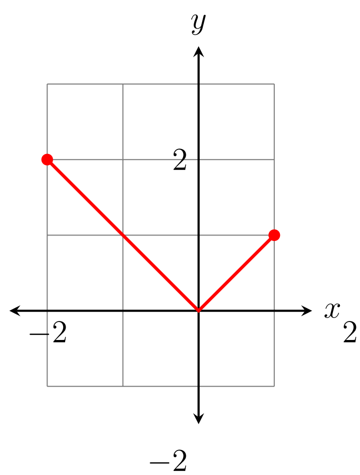



Figure 7: The graph of $f(x) = |x|$ on the closed interval $[-2, 1]$.

Now let's find the global extrema.

1. Since the pieces of this function are lines with slopes $-1$ and 1, the derivative is never equal to 0. But as we saw before, the function $f(x)$ is *not* differentiable at $x = 0$. So we must include $x = 0$ as well as the endpoints.

2. Evaluating the function:

$$f(0) = 0, \quad f(-2) = 2, \quad f(1) = 1.$$

3. The lowest function value is 0, so there is a global minimum at $(0, 0)$. The highest function value is 2, so there is a global maximum at $(-2, 2)$.

## Example 6

Let's look at another example where we need to look at where $f'(x)$ doesn't exist. Consider $f(x) = \sqrt{x}$ on the closed interval $[0, 4]$, shown in Figure 8.

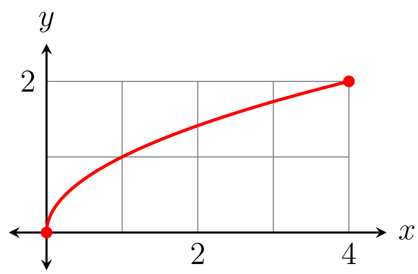



Figure 8: The graph of $f(x) = \sqrt{x}$ on the closed interval $[0, 4]$.

Again, let's apply the steps.

1. Writing $f(x) = x^{1/2}$, we use the Power Rule to get

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

   Note that the derivative can never be 0, because the numerator is always 1. But there *is* a place the derivative is undefined: $x = 0$. This is because you can't have a 0 in the denominator. What this means on the graph is that there is a vertical tangent at $x = 0$, and we know that vertical lines have an undefined slope.

2. Evaluating the function:

$$f(0) = 0, \quad f(4) = 2.$$

3. Looking for lowest and highest values, we have a global minmum at $(0, 0)$ and a global maximum at $(4, 2)$. It turns out that the only place $f'(x)$ is undefined is at an endpoint, but that won't always be the case.

## Homework

1. Find the local extrema for the function $f(x) = x - 2\cos(x)$ on the interval $[0, 2\pi]$. Check that you're right by graphing.

2. Find the local extrema for the function $f(x) = x^5$ using the method in the notes. We can look at a graph and see that there are none, but use calculus to show it.

3. Find the global extrema for the function $f(x) = x^{2/3}$ on the closed interval $[-4, 4]$. Graph this function on desmos (or your calculator) to verify your answer.

4. Find the global extrema for the function $f(x) = e^x - x$ on the closed interval $[-5, 2]$.

5. Find the global extrema for the function $f(x) = x - \frac{1}{4}\ln x$ on the closed interval $[1, 7]$.

## Solutions

1. (a) We first find where $f'(x) = 0$ using the unit circle.

$$\begin{aligned} f(x) &= x - 2\cos(x) \\ f'(x) &= 1 + 2\sin(x) = 0 \\ \sin(x) &= -\frac{1}{2} \\ x &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

   (b) Next, we evaluate $f''(x)$ at these points.

$$\begin{aligned} f'(x) &= 1 + 2\sin(x) \\ f''(x) &= 2\cos(x) \end{aligned}$$

Since $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} < 0$, there is a local maximum at $x = \frac{7\pi}{6}$. Since $\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} > 0$, there is a local minimum at $x = \frac{7\pi}{6}$.

2. (a) We first find where $f'(x) = 0$.

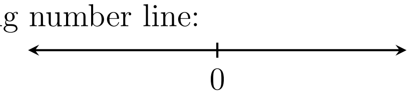
$$\begin{aligned} f(x) &= x^5 \\ f'(x) &= 5x^4 \\ 5x^4 &= 0 \\ x &= 0 \end{aligned}$$

   (b) Now check $f''(x)$.

$$\begin{aligned} f'(x) &= 5x^4 \\ f''(x) &= 20x^3 \\ f''(0) &= 0 \end{aligned}$$

Since $f''(0) = 0$, we need to make a sign chart.

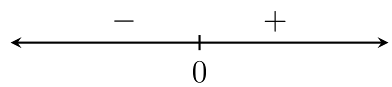
   i. We already know that solving $f''(x) = 0$ gives $x = 0$.
   ii. This gives the following number line:



iii. Now choose one value from each interval. Easy values are $x = -1$ and $x = 1$.

$$\begin{aligned} f''(-1) &= 20(-1)^3 \\ &= -20 \\ &< 0 \\ f''(1) &= 20(1)^3 \\ 20 \\ &> 0. \end{aligned}$$

This yields the following number line:




Thus, since the concavity changes at $x = 0$, there must be an inflection point there, and therefore no local extrema exist.

3. (a) First, we determine when $f'(x) = 0$ or when $f'(x)$ does not exist.

$$\begin{aligned} f(x) &= x^{2/3} \\ f'(x) &= \frac{2}{3}x^{-1/3} \\ &= \frac{2}{3\sqrt[3]{x}} \end{aligned}$$

$f'(x)$ can never be 0 since the numerator cannot be 0. $f'(x)$ is undefined when $x = 0$ since you can't have 0 on the denominator.

(b) Now evaluate at this point and the endpoints.

$$f(0) = 0, \quad f(-4) = \sqrt[3]{16}, \quad f(4) = \sqrt[3]{16}.$$

(c) Looking at largest and smallest values, we have a global minimum at $(0, 0)$ and global maxima at $(-4, \sqrt[3]{16})$ and $(4, \sqrt[3]{16})$.

4. (a) First, we determine when $f'(x) = 0$ or when $f'(x)$ does not exist.

$$\begin{aligned} f(x) &= e^x - x \\ f'(x) &= e^x - 1 = 0 \\ e^x &= 1 \\ x &= 0 \end{aligned}$$

$f'(x)$ always exists since $e^x$ exists for every $x$.

(b) Now evaluate at this point and the endpoints.

$$f(0) = e^0 - 1 = 0, \quad f(-5) = e^{-5} - (-5) \approx 4.99, \quad f(2) = e^2 - 2 \approx 5.39.$$

(c) Looking at largest and smallest values, we have a global minimum at $(0, 0)$ and global maximum at $(2, e^2 - 2)$.

5. (a) First, we determine when $f'(x) = 0$ or when $f'(x)$ does not exist.

$$\begin{aligned} f(x) &= x - \frac{1}{4}\ln x \\ f'(x) &= 1 - \frac{1}{4} \cdot \frac{1}{x} = 1 - \frac{1}{4x} = 0 \\ \frac{1}{4x} &= 1 \\ 4x &= 1 \\ x &= \frac{1}{4} \end{aligned}$$

However, we cannot consider this point since $\frac{1}{4}$ is *not* in the interval $[1, 7]$.

$f'(x)$ always exists since the denominator cannot be 0 since we are looking at the closed interval $[1, 7]$.

(b) Now evaluate at the endpoints.

$$f(1) = 1 - \frac{1}{4}\ln 1 = 1, \quad f(7) = 7 - \frac{1}{4}\ln 7 \approx 6.51$$

(c) Looking at largest and smallest values, we have a global minimum at $(1, 1)$ and global maximum at $\left(7, 7 - \frac{1}{4}\ln 7\right)$.