Limits and Continuity

Up to this point, we have used limits to help us define derivatives. The reason we needed the idea of a limit is that we looked at the slopes of secant lines, such as

$$m = \frac{f(x+h) - f(x)}{h}.$$

We wanted to see what happens as $h \to 0$, but we can't just plug in $h = 0$. Or else we'd get $\frac{0}{0}$, which is undefined. So we wrote

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Limits have many other uses in mathematics and calculus. We'll focus here on the concept of **continuity,** which essentially means there are no "jumps" in the graph of a curve.

Example 1

We first look at a function $f(x)$, where x is the number of ounces your letter weighs and $f(x)$ is how much is costs to send your letter. The graph is shown in Figure 7. If you mail a first class letter, you pay \$0.60 for the first ounce (up to exactly one ounce), and \$0.24 for each part of an ounce after that.

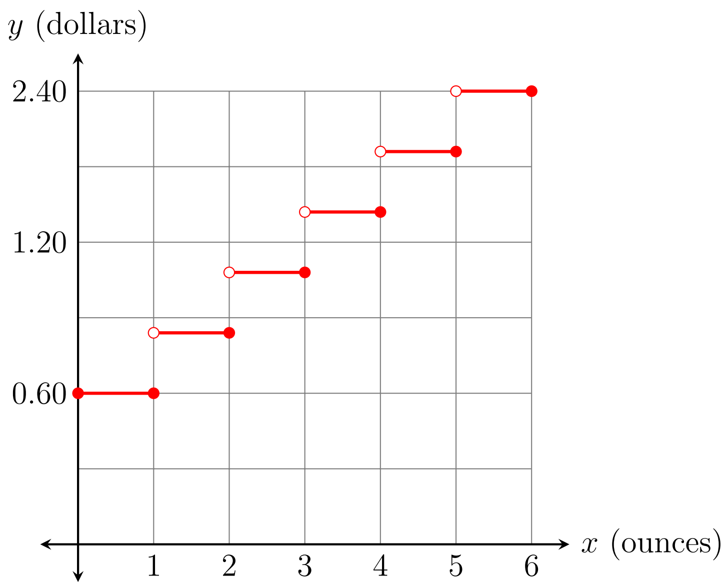


Figure 1: Graph of 2022 postal rates.

Notice the jumps. If your letter is exactly one ounce, you pay \$0.60 to mail it. But if it's the slightest bit over, you pay \$0.84. There is no letter which will ever cost any other price between \$0.60 and \$0.84.

How do we describe this using limits? We will introduce the concepts of **left-handed limits** and **right-handed limits.**

In Figure 2, we have zoomed in on a part of the graph near $x = 1$. Notice there is a thin strip blocking out the part of the graph right at $x = 1$. As we saw in calculating derivatives, you can't look at $h = 0$ right away, since it's undefined. So in looking at limits, we look *around* an x-value and see what's happening.

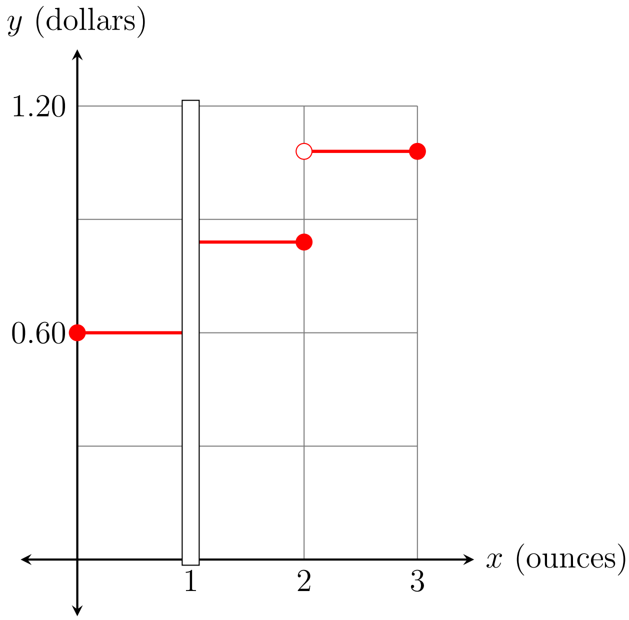


Figure 2: Graph of 2022 postal rates (closeup).

Start by looking at what happens when x moves to 1 coming from the left. It looks like at $x = 1$, the value of the function is 0.60. But if we start a little bit to the right of $x = 1$ and move *left,* it looks like the value of the function is 0.84. (Of course it can't be *both,* since one input, $x = 1$, cannot have two outputs.) In the language of limits, we say

$$\lim_{x \to 1^-} f(x) = 0.60, \quad \lim_{x \to 1^+} f(x) = 0.84.$$

We read these as "the limit as x approaches 1 from the left of $f(x)$ is 0.60," and "the limit as x approaches 1 from the right of $f(x)$ is 0.84." In general, the "$-$" as a superscript means looking from the left, and the "+" means looking from the right.

Now we know that $f(1) = 0.60$ from Figure 7. So

$$\lim_{x \to 1^-} f(x) = 0.60, \quad f(1) = 0.60, \quad \lim_{x \to 1^+} f(x) = 0.84.$$

This is how, using the language of limits, we can say that there is a jump in a graph.

Example 2

Let's take a look at another example with a jump in the graph.

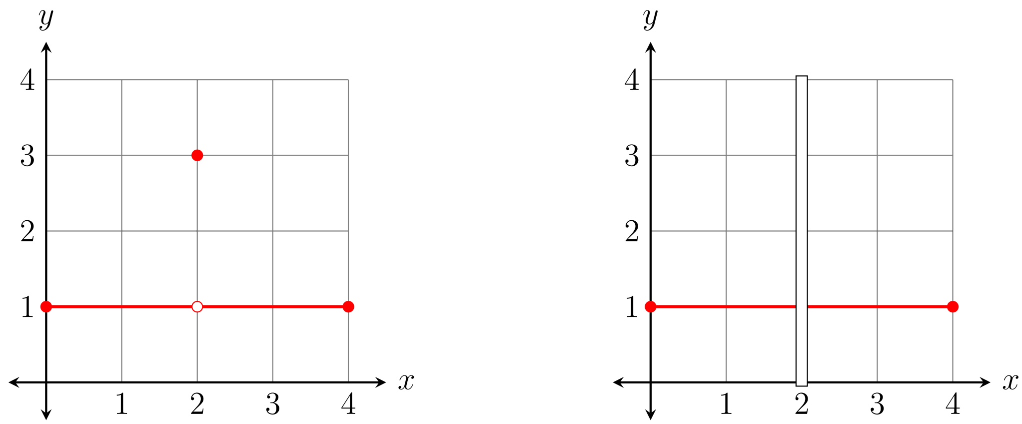


Figure 3: Graph of a function $g(x)$ with a removable discontinuity.

What is happening at $x = 2$? Looking at the right graph in Figure 3, we can see that

$$\lim_{x \to 2^-} g(x) = 1 = \lim_{x \to 2^+} g(x).$$

When both left-hand and right-hand limits are the same, we can simply say

$$\lim_{x \to 2} g(x) = 1.$$

But $g(2) = 3 \neq 1$, so there is a discontinuity at $x = 2$. We say this type of discontinuity is a **removable discontinuity** since we can redefine $g(x)$ at 2 to make it continuous at $x = 2$. If we make $g(2) = 1$, the function would be continuous at $x = 2$.

Note the difference at $x = 1$ in Figure 2. It doesn't matter how we define $f(1)$, there *must* be a jump. In this case, we say that $x = 1$ is an **essential discontinuity.**

Example 3

Let's take a look an example without a jump. In this case, $h(x) = x + 1$.

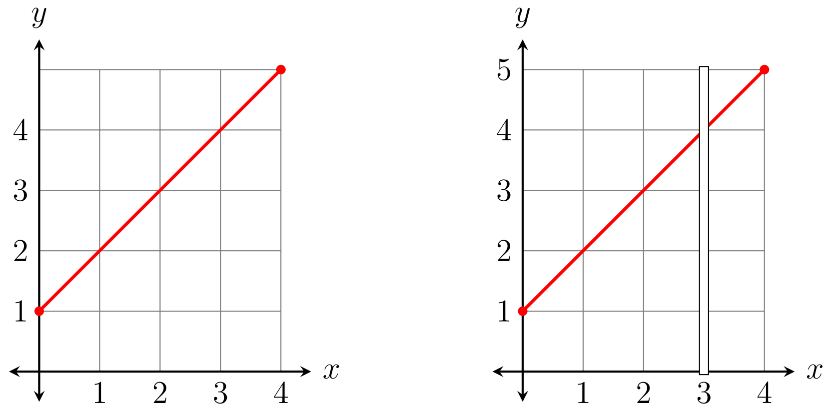


Figure 4: Graph of a function $h(x)$ without a jump.

Let's look at what is happening at $x = 3$. Looking at the right graph in Figure 4, we can see that

$$\lim_{x\to 3^-} h(x) = 4 = \lim_{x\to 3^+} h(x).$$

Since both left-hand and right-hand limits are the same, we can say

$$\lim_{x\to 3} h(x) = 4.$$

But $h(3) = 4$ as well, so we can say that $h(x)$ is **continuous** at $x = 3$.

If a function is continuous at *every* point in its domain, we say **the function is continuous.** If we are only looking at a single point, we just say that a function is continuous at a particular point, as in this example.

Here is a summary of the new terminology we encountered in the first three examples.

Let $f(x)$ be given and let a be in its domain.

If...	then...
$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+}$	$f(x)$ has an essential discontinuity at a
$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} \neq f(a)$	$f(x)$ has a removable discontinuity at a
$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} = f(a)$	$f(x)$ is continuous at a

It is important to note that the limits mentioned in this chart do not always exist for every function. We will look at such examples later; for now, we'll stick to functions where these limits exist.

Example 4

Most of the examples we'll come across will be continuous. Proving that a function is continuous can often involve a lot of work with limits. To avoid all these proofs, we'll summarize the main points below.

The following functions are continuous wherever they are defined:

1. Polynomials, such as $f(x) = 3x^5 - 4x^2 + 7$,
2. Roots/radicals, such as $f(x) = x^{2/3}$,
3. Rational functions, such as $f(x) = \dfrac{3x^2 - 1}{x + 4}$,
4. The basic trigonometric functions: $\sin(x)$, $\cos(x)$, and $\tan(x)$,
5. $f(x) = e^x$,
6. $f(x) = \ln x$.

If $f(x)$ and $g(x)$ are continuous functions, the following are also continuous:

7. $c \cdot f(x)$, where c is a constant,
8. $f(x) + g(x)$,
9. $f(x) - g(x)$,
10. $\dfrac{f(x)}{g(x)}$, as long as $g(x) \neq 0$,
11. $(f \circ g)(x)$.

What this means is that it's very easy to create new continuous functions from the basic ones. To see an example, let's show that $h(x) = \sin(x^2 + 1)$ is a continuous function. First, we write $h(x)$ as a function composition. With $f(x) = \sin(x)$ and $g(x) = x^2 + 1$, we have $h(x) = (f \circ g)(x)$. But $f(x)$ is continuous by (4) above and $g(x)$ is continuous by (1) above. Then by (11), their composition is continuous, which is just $h(x)$.

Example 5

Let's look now at $f(x) = \frac{1}{x}$. Note that here, the domain is all real numbers *except* $x = 0$. By (3) above, we know that $f(x)$ is a continuous function. But how does that make sense? Doesn't the graph make a huge jump across $x = 0$?

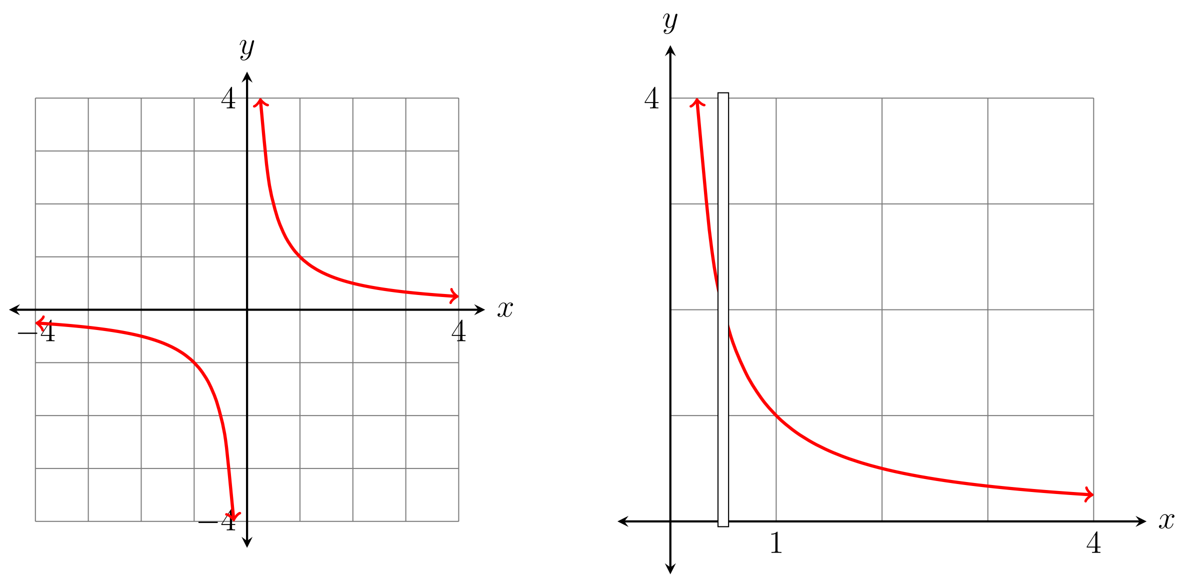


Figure 5: Graph of $f(x) = \frac{1}{x}$ (left), closeup at $x = \frac{1}{2}$ (right).

The important concept here is that you can only talk about continuity where a function is *defined.* Look at the right graph in Figure 5. Is $f(x)$ continuous at $x = \frac{1}{2}$? You should convince yourself that

$$\lim_{x\to 1/2^-} f(x) = \lim_{x\to 1/2^+} f(x) = f\left(\frac{1}{2}\right) = 2.$$

This means precisely that $f(x)$ is continuous at $\frac{1}{2}$. We can draw a similar thin vertical strip and make similar observations at any point on the graph.

But we *cannot* draw a vertical strip at $x = 0$, since $f(0)$ is *undefined.* This seems like we're being a bit picky, but this concept is *very* important to understanding continuity. It will be especially important when we explore asymptotes of functions.

Homework

1. For each of the terms in bold face in this section, be prepared to write a once-sentence definition in your own words.

2. Suppose a function $f(x)$ is defined for all real numbers. You know that

$$\lim_{x\to 2^-} f(x) = 3, \quad \lim_{x\to 2^+} f(x) = 4.$$

   Which of the following are possible? Circle all that apply.

   (a) $f(x)$ is continuous at $x = 2$.

   (b) $f(x)$ has an essential discontinuity at $x = 2$.

   (c) $f(x)$ has a removable discontinuity at $x = 2$.

3. Suppose a function $f(x)$ is defined for all real numbers. You know that

$$\lim_{x\to -1^+} f(x) = f(-1).$$

   Which of the following are possible? Circle all that apply.

   (a) $f(x)$ is continuous at $x = -1$.

   (b) $f(x)$ has an essential discontinuity at $x = -1$.

   (c) $f(x)$ has a removable discontinuity at $x = -1$.

4. On the grid below, sketch a graph of a function $f(x)$ which has the following properties. Note: many answers are possible; there is not just one correct answer.

(a) There is a removable discontinuity at $x = 1$.

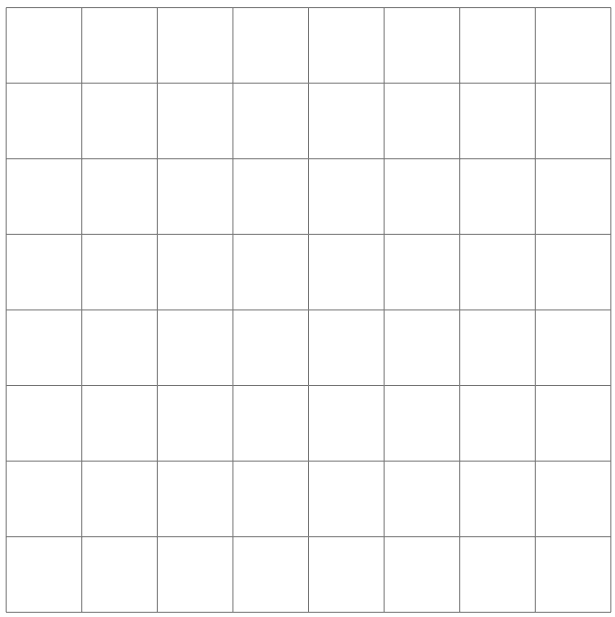
(b) There is an essential discontinuity at $x = 3$.

(c) $\lim_{x \to -1^-} f(x) = -2$.

(d) $\lim_{x \to 3^+} f(x) = 2$.

(e) $f(-2) = -1$.

(f) $f(x)$ is continuous at $x = -2$.



5. Suppose $f(x) = \dfrac{1}{x^2 - 1}$. The graph is shown below.

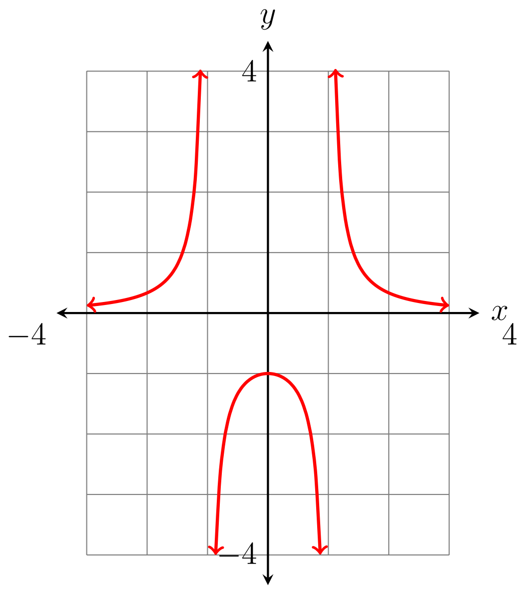


Figure 6: Graph of $f(x) = \dfrac{1}{x^2 - 1}$.

You are discussing with your friend whether or not this is a continuous function. She says, “No way! Look at how the function jumps around! It looks like there are vertical asymptotes. This function can’t be continuous!”

What is your repsonse to your friend?

6. The function $f(x) = \lceil x \rceil$ is called the *ceiling function*. It is defined so that $f(x)$ is the smallest integer that is greater than or equal to x. Thus,

$$\lceil 4 \rceil = 4, \quad \lceil 4.5 \rceil = 5, \quad \lceil -2 \rceil = -2, \quad \lceil -1.5 \rceil = -1.$$

Part of its graph is shown below.

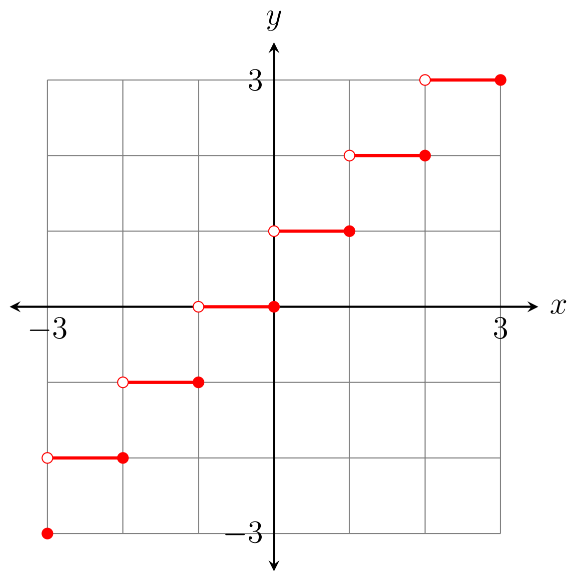


Figure 7: Partial graph of the ceiling function.

(a) What is $f(-4.8)$? $f(4.8)$?

(b) Describe the behavior of the graph at $x = 0$ using the notations and terminology of this section.

(c) Choose the best answer. At $x = 0.5$, the function:

i. Is continuous.

ii. Has an essential discontinuity.

iii. Has a removable discontinuity.

7. Suppose you are given the following graph, with a thin vertical strip covering $x = a$. Which of the following are possible? Circle all that apply.

(a) $f(x)$ is continuous at $x = a$.

(b) $f(x)$ has an essential discontinuity at $x = a$.

(c) $f(x)$ has a removable discontinuity at $x = a$.

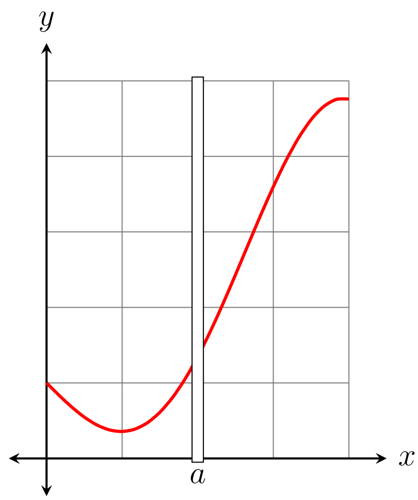


Figure 8: Graph of a function $f(x)$.

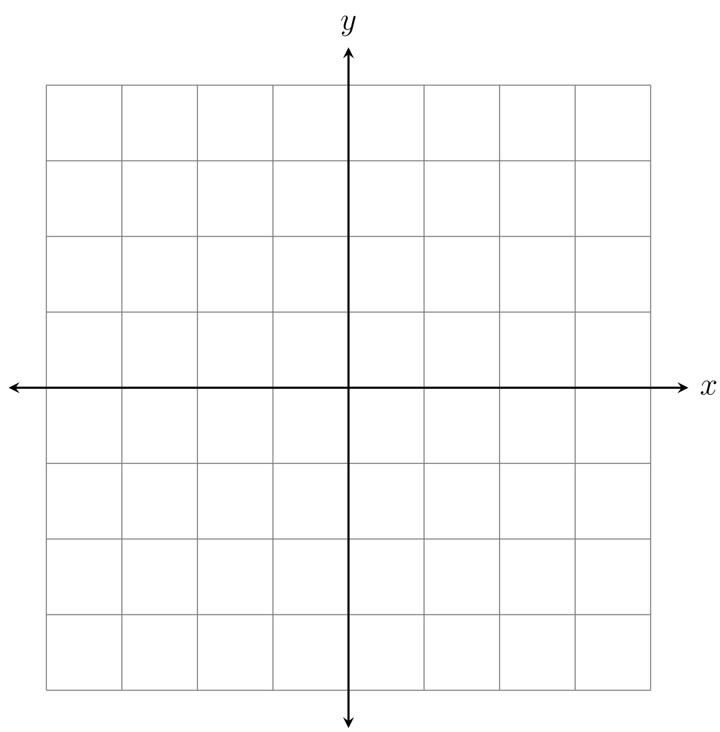
8. Sometimes it makes more than one formula to define a function. You might have seen the absolute value function, $f(x) = |x|$, defined in a **piecewise** way:

$$f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Consider the following piecewise-defined function. Assume b is a constant.

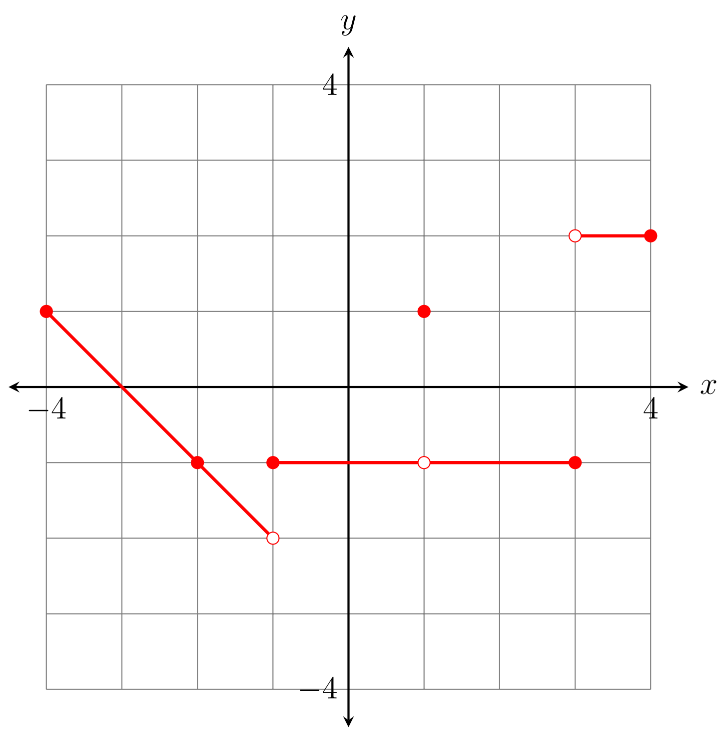
$$g(x) = \begin{cases} x + 1, & x \leq 1, \\ b - x, & x > 1. \end{cases}$$

What must be the value of b so that $g(x)$ is a continuous function? Sketch a graph of this function on the interval $[-4, 4]$ below.



Solutions

1. Answers will be different for everyone.
2. (b).
3. (a) and (b).
4. Many other answers are possible. This is just one way to satisfy all the given properties.



5. You would tell your friend to be very careful with the definition of continuity! Continuity can only be discussed where a function is *defined.* Since the function is *not* defined at $x = -1, 1$, these values cannot detract from the continuity of the function. Wherever the function *is* defined, it is continuous. (See Example 5 for another example.)
6. (a) $f(-4.8) = -4$, $f(4.8) = 5$.

   (b)
   $$\lim_{x \to 0^-} f(x) = 0, \quad \lim_{x \to 0^+} f(x) = 1.$$
   Therefore, there is an essential discontinuity at $x = 0$.

   (c) (i).
7. (a), (c).

8. $g(x)$ has pieces which are lines (which are polynomials), so we have to check that they "meet up" at $x = 1$. Keeping in mind that b is a constant, we have

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} (x + 1) = 2$$

and

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (b - x) = b - 1.$$

To be continuous at $x = 1$, these left-hand and right-hand limits must be equal, and so

$$2 = b - 1.$$

This gives $b = 3$. We graph the function $x + 1$ to the left of $x = 1$ and the function $3 - x$ to the right of $x = 1$. We chose b so that they meet up perfectly at $x = 1$.