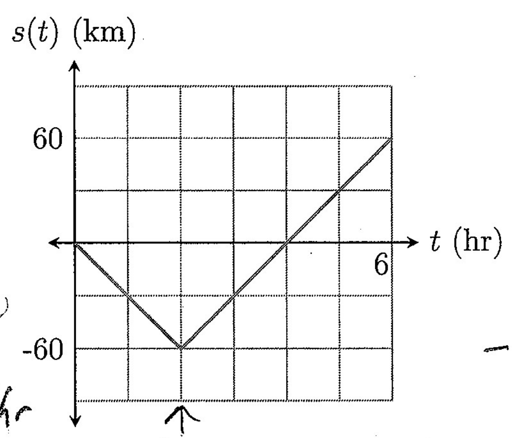
Exam 1, MA 131M, 10 October 2022

Please show as much work as you need. Keep in mind that if you skip several steps and get an incorrect answer, it will be very difficult to assign partial credit. Please take a moment to skim through the test first and start with the questions you feel most confident about. Your answers do NOT have to be in order on your paper.

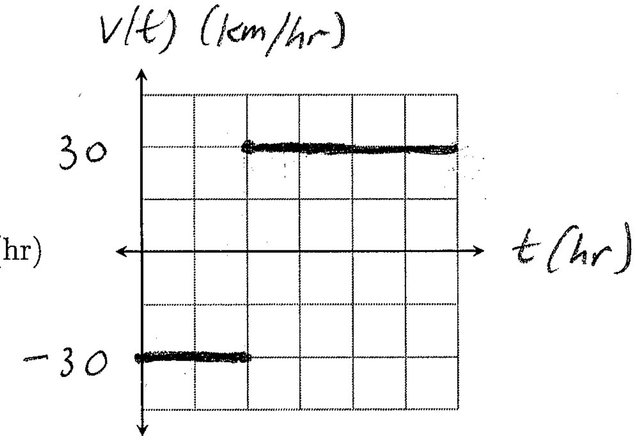
1. (15) Using the **definition of a derivative,** find $f'(x)$ if $f(x) = 4x - x^2$.

2. (12) A graph of a displacement curve is shown on the left. Draw the corresponding velocity curve on the blank graph on the right. Label axes completely and carefully!

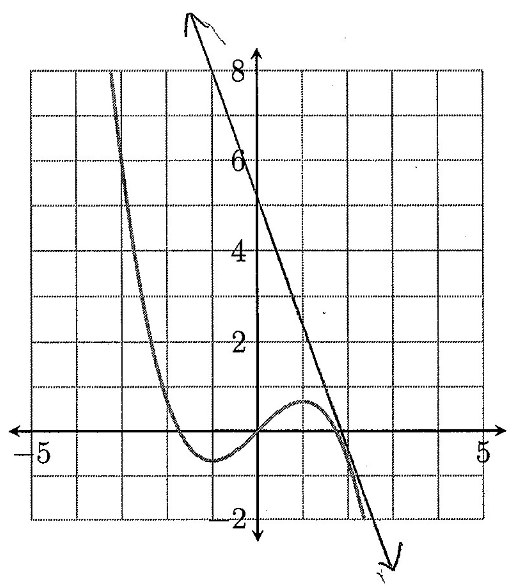
$\frac{-60 \text{ km}}{2 \text{ hr}}$ $= -30$ km/hr



t=2



3. (12) Find the equation of the line tangent to $y = x - \frac{1}{3}x^3$ at $x = 2$. The graph is given below. Check your work by sketching the tangent line on the graph.



Equation of tangent line, $y = -3x + \frac{16}{3}$, visually checks out.

4. (10) The population of a colony of bacteria at time t (in hours) is given by the equation $P(t) = 2000e^{0.025t}$. Find out how fast the colony is growing at time $t = 4$.

5. Take the derivatives of the following functions:
   (a) (5) $h(x) = \dfrac{4}{x^5} - \sqrt{x^7}$
   (b) (7) $h(x) = \dfrac{a-3x}{a+3x}$
   (c) (7) $h(x) = x^2 \cos(x)$

6. (10) Suppose you are given $f(x) = 4x - \frac{1}{3}x^3$, so that $f'(x) = 4 - x^2$. Using a **sign chart,** determine where the function is increasing and decreasing. Write your answer in **interval notation.**

7. (12) Suppose you are given the following function and its derivatives:

$$\begin{aligned} f(x) &= \frac{1}{4}x^4 + x^3 - 4x \\ f'(x) &= x^3 + 3x^2 - 4 = (x-1)(x+2)^2 \\ f''(x) &= 3x^2 + 6x \end{aligned}$$

   Determine where $f'(x) = 0$. Using a **sign chart,** determine if there is a minimum, maximum, or inflection point at these values of x.

8. Fill in the blank with the best answer.
   (a) (3) If $f'(x) > 0$, then the function is increasing at this value of x.
   (b) (3) If $f''(x) < 0$, then the function is concave down at this value of x.

9. (2) TRUE FALSE If $f(x)$ is concave up at x, then $f'(x) > 0$.

10. (2) TRUE FALSE If $f''(x) = 0$, there must be an inflection point at x.

EXTRA CREDIT: Using the **definition of the derivative,** find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt{x}}$.

SOLUTIONS, Exam I

1. $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{4(x+h)-(x+h)^2-(4x-x^2)}{h}$

$= \lim_{h\to 0} \frac{\cancel{4x}+4h-\cancel{x^2}-2xh-h^2-\cancel{4x}+\cancel{x^2}}{h}$

$= \lim_{h\to 0} \frac{4h-2xh-h^2}{h} = \lim_{h\to 0} \frac{h(4-2x-h)}{h}$

$= \lim_{h\to 0} (4-2x-h) = 4-2x$

3. $f'(x) = 1-x^2$

$f'(2) = 1-2^2 = -3$, $m = -3$ (slope of tangent line)

Point on line: $(2, f(2)) = (2, 2-\frac{8}{3}) = (2, -\frac{2}{3})$

$y - y_1 = m(x-x_1)$

$y-(-\frac{2}{3}) = -3(x-2) = -3x+6$

$y = -3x+6-\frac{2}{3} = -3x+\frac{16}{3}$

4. $P(t) = 2000e^{0.025t}$

$f(t) = 2000e^t$ $f'(t) = 2000e^t$

$g(t) = 0.025t$ $g'(t) = 0.025$

$P'(t) = f'(g(t))\,g'(t)$

$= 2000\,e^{g(t)} \cdot 0.025$

$= 50\,e^{0.025t}$

At $t=4$: $P'(4) = 50\,e^{0.025(4)} \approx 55.3$.

The colony is growing at about 56 bacteria/hr at time $t=4$.

SOLUTIONS, Exam I

5(a) $h(x) = \frac{4}{x^5} - \sqrt{x^7} = 4x^{-5} - x^{7/2}$

$h'(x) = -20x^{-6} - \frac{7}{2}x^{5/2}$

(b) $h(x) = \frac{a-3x}{a+3x} \begin{matrix} \leftarrow f(x) \\ \leftarrow g(x) \end{matrix}$

$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x))^2}$

$= \frac{(a+3x)(-3) - (a-3x)(3)}{(a+3x)^2} = \frac{-3a - 9x - 3a + 9x}{(a+3x)^2}$

$= \frac{-6a}{(a+3x)^2}$

(c) $h(x) = x^2\cos(x)$ $\quad f(x) = x^2 \quad f'(x) = 2x$

$\quad g(x) = \cos(x) \quad g'(x) = -\sin(x)$

$h'(x) = f(x)g'(x) + g(x)f'(x)$

$= x^2 \cdot (-\sin(x)) + \cos(x) \cdot 2x$

$= -x^2\sin(x) + 2x\cos(x)$

6. $f'(x) = 4 - x^2 = 0$

$x^2 = 4$

$x = \pm 2 \rightarrow$

$-$ $\quad$ $+$ $\quad$ $-$ $\quad f'(x)$

$\longleftarrow$ -2 $\quad$ 2 $\longrightarrow$

$-3 \quad 0 \quad 3$ ←Test points

$f'(-3) = 4 - (-3)^2 = -5 < 0$

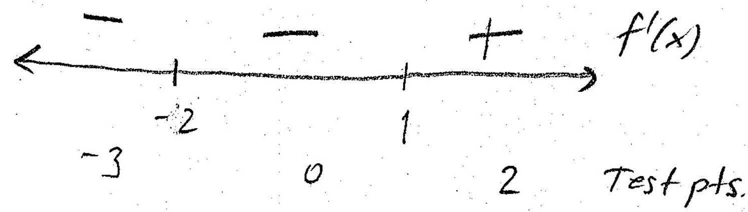
$f'(0) = 4 - 0^2 = 4 > 0$

$f'(3) = 4 - 3^2 = -5 < 0$

$f(x)$ is increasing on $(-2, 2)$ and decreasing on $(-\infty, -2) \cup (2, \infty)$.

7. $f'(x) = (x-1)(x+2)^2 = 0$

$x = 1, \quad x = -2$



$f'(-3) = (-3-1)(-3+2)^2 = -4 < 0$

$f'(0) = (0-1)(0+2)^2 = -4 < 0$

$f'(2) = (2-1)(2+2)^2 = 16 > 0$

Since we decrease through $x = -2$, there is an inflection point. Since we change from decreasing to increasing through $x = 1$, there is a local minimum there.

Extra Credit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$ get a common denominator

$$= \lim_{h \to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{h} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}$$ multiply by the conjugate

$$= \lim_{h \to 0} \frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{x + \sqrt{x}\sqrt{x+h} - \sqrt{x+h}\sqrt{x} - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \lim_{h \to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x} \cdot \sqrt{x}(2\sqrt{x})} = \frac{-1}{2x^{3/2}}$$